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A MODEL UNIVERSE WITH VARIABLE SPACE DIMENSION: ITS DYNAMICS AND WAVE FUNCTION

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Abstract

Assuming the space dimension is not constant, but varies with the expansion of the universe, a Lagrangian formulation of a toy universe model is given. After a critical review of previous works, the field equations are derived and discussed. It is shown that this generalization of the FRW cosmology is not unique. There is a free parameter in the theory, C , with which we can fix the dimension of space say at the Planck time. Different possibilities for this dimension are discussed. The standard FRW model corresponds to the limiting case $C \rightarrow +\infty$. Depending on the free parameter of the theory, C , the expansion of the model can behave differently to the standard cosmological models with constant dimension. This is explicitly studied in the framework of quantum cosmology. The Wheeler–De Witt equation is written down. It turns out that in our model universe, the potential of the Wheeler–DeWitt equation has different characteristics relative to the potential of the de Sitter minisuperspace. Using the appropriate boundary conditions and the semiclassical approximation, we calculate the wave function of our model universe. In the limit of $C \rightarrow +\infty$, corresponding to the case of constant space dimension, our wave function has not a unique behavior. It can either leads to the Hartle–Hawking wave function or to a modified Linde wave function, or to a more general one, but not to that of Vilenkin. We also calculate the probability density in our model universe. It is always more than the probability density of the de Sitter minisuperspace in 3-space as suggested by Vilenkin, Linde, and others. In the limit of constant space dimension, the probability density of our model universe approaches to Vilenkin and Linde probability density being $\exp(-2|S_E|)$, where S_E is the Euclidean action. Our model universe indicates therefore that the Vilenkin wave function is not stable with respect to the variation of space dimension.

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1 INTRODUCTION

Kaluza–Klein and string theories are well known for allowing the space dimension to be other than 3, but still an integer and a constant [1, 2]. This, being considered for the high energy limit in the universe or for the dimension of space at the Planck time, has encouraged people to suggest that the dimension of space in the lower energy limit, or for the actual structured universe, is other than 3. Recently models have been proposed in which the universe has $(1 + 3 + n)$ -dimensional space with Planck scale near the weak scale, with $n \geq 2$ new sub-millimeter sized dimensions [3].

In the last years there has been attempts to identify a fractal dimension for the matter distribution in space using either the cosmic microwave background radiation or the galaxy distributions [4, 5]. Aside from the actual dimension of space or the matter distribution in it, it is interesting to study the cosmological consequences of a fractal and variable space dimension. All critiques of space dimensionality other than 3 rely upon cosmologically small scale observations [6]. Therefore, one could ask about the consequences of a dynamical space dimension in cosmological time and space scales. A proposed way of handling such a concept is using the idea of decrumpling coming from the polymer physics [7–9]. The evolution of the fractal dimension of a self-similar universe in the context of Newtonian gravitation is studied in [10].

Our toy model universe consists of small cells having arbitrary space dimensions. The effective spatial dimension of the universe in large will depend on the configuration of the cells. As an example, take a limited number of small three dimensional beads. Depending on how these beads are embedded in space they can configure to a one-dimensional string, 2-dimensional sheet, or a three dimensional sphere. This is the picture we are familiar from the concept of crumpling in the polymer physics where a crumpled polymer has a dimension more than 1. Or take the picture of a clay which can be like a 3 dimensional sphere, or a 2 dimensional sheet, or even a one dimensional string; a picture we are familiar from the theory of the fluid membranes. In this picture the universe can have any space dimension. As it expands the space dimension decreases continuously.

In this paper we modify and study thoroughly the model proposed in [11–13]. The original decrumpling model of the universe seems to be singularity free having two turning points for the space dimension [11, 12]. Lima *et al.* [13] criticize the way of generalizing the standard cosmological model to arbitrary space dimension used in [11] and propose another way of writing the field equations. Their model shows no upper bound for the dimension of space. As the universe expands the spatial dimension decreases to $D = 1$ [13].

Later on this scenario was extended to the class of multidimensional cosmological models, where extra factor spaces play the role of the matter fields. In this multidimensional cosmological model an inflationary solution was found together with the prediction that the universe starts from a nonsingular space time [14]. Melnikov *et al.* has written the Wheeler–DeWitt equation for multidimensional cosmological model in any constant spatial dimension [15], see also [16] and references therein.

We review briefly the previous works on the decrumpling models and show some difficulties in their formulations and conclusions. To remove these difficulties, we propose a new way to generalize the gravitational action in constant dimension to the case of dynamical dimension. We use the Hawking–Ellis action for a perfect fluid in 3-space [17] and generalize it to the case of dynamical dimension. As we will show in details the generalization of the gravitational action to the dynamical dimension is not unique. Moreover, in contrast to the earlier

works in [11–13], we take into account the dependence of the measure of the action on space dimension. The generalization of the action, the Lagrangian, and the equations of motion to dynamical space dimension is then done in two ways. Studying the time evolution of the spatial dimension, we obtain some numerical results for the turning points in our model.

In the second part of the paper we study the quantum cosmology of our model universe. The Wheeler–DeWitt equation is written and the zero points of its potential are discussed. It is worth noticing that the potential of our model has completely different behavior from the potential of the de Sitter minisuperspace in 3-space. Imposing the appropriate boundary condition in the limit $a \rightarrow +\infty$, and using the semiclassical approximation, we obtain the wave function of our model. It is then seen that in the limit of constant space dimension, our wave function is not well-defined. We show that it can approach to the Hartle–Hawking wave function or to the modified Linde wave function, but not to that of Vilenkin. We also estimate the probability density in our model universe. In the limit of constant spatial dimension, it approaches to Vilenkin, Linde and others’ proposal [18]; i.e. to the probability density $\mathcal{P} \propto \exp(2S_E)$, or more generally $\exp(-2|S_E|)$, where S_E is the Euclidean action of the classical instanton solution.

The paper is organized as follows. In Sec. 2 we discuss the dimensional constraint, proposed in [11]. In Sec. 3, we briefly review the earlier works on the decrumpling model and discuss their shortcomings. In Sec. 4, a new way to generalize the gravitational- and the matter-action to the case of dynamical dimension is presented. The equation of motion and the time evolution equation of spatial dimension is then obtained. Sec. 5 is devoted to the Wheeler–DeWitt equation of our model and the calculation of its wave function in semiclassical approximation. The probability density in our model universe is then estimated semiclassically. Section 6 concludes the discussion of our toy model universe.

2 MOTIVATION AND CONSTRAINT OF THE MODEL

There are observational evidences that the matter distribution in the universe, up to the present observed limits of $100h^{-1}\text{Mpc}$, is a fractal or multifractal [4, 5] having a dimension of about 2. This should be considered as an effective dimension of space without questioning the three dimensionality at cosmologically small scales [6].

To interpret the effective space dimension D we follow the picture proposed in [11]–[13]. Imagine the fundamental building blocks of the universe as cells being arbitrary dimensional and having, in each dimension, a characteristic size δ which maybe of the order of the Planck length $\mathcal{O}(10^{-33}\text{cm})$ or even smaller. These “space-cells” are embedded in a \mathcal{D} space, where \mathcal{D} may be up to infinity. Therefore, the space dimension of the universe depends on how these fundamental cells are configured in this embedding space. The universe may have begun from a very crumpled state having a very high dimension \mathcal{D} and a size δ , then have lost dimension through a uniform decrumpling which we see it as a uniform expansion. The expansion of space, being now understood as a decrumpling of cosmic space, reduce the space-time dimension continuously from $\mathcal{D} + 1$ to the present value $D_0 + 1$. We do not fix D_0 to allow being other than 3.

Here, we are interested in the correlation between the radius of our model universe and its dimension. In the next section, we will implement this idea in the Lagrangian formulation. The first major formal difficulty in formulating a space-time theory with variable space dimension is that the measure of the integral action is variable and therefore some part of

integrand. The variational calculus for such a case has not been formulated yet. We are, however, interested in a cosmological model for the actual universe. Therefore, it is reasonable to accept the cosmological principle: the homogeneity and isotropy of space. This simplifies the matter substantially, so that it becomes possible to formulate a Lagrangian for the theory and write the corresponding field equations.

2.1 Relation between the effective space dimension $D(t)$ and characteristic size of the universe $a(t)$

Assume the universe consists of a fixed number N of universal cells having a characteristic length δ in each of their dimensions. The volume of the universe at the time t depends on the configuration of the cells. It is easily seen that [11, 12]

$$\text{vol}_D(\text{cell}) = \text{vol}_{D_0}(\text{cell})\delta^{D-D_0}. \quad (1)$$

Interpreting the radius of the universe, a , as the radius of gyration of crumpled “universal surface” [9], the volume of space can be written [11, 12]:

$$\begin{aligned} a^D &= N \text{vol}_D(\text{cell}) \\ &= N \text{vol}_{D_0}(\text{cell})\delta^{D-D_0} \\ &= a_0^{D_0} \delta^{D-D_0}, \end{aligned} \quad (2)$$

or

$$\left(\frac{a}{\delta}\right)^D = e^C, \quad (3)$$

where C is a universal positive constant. Its value has a strong influence on the dynamics of space-time, for example on the dimension of space say at the Planck time. Hence, it has physical and cosmological consequences and may be determined by observation. We coin the above relation as “dimensional constraint” which relates the “scale factor” of our model universe to the space dimension. Aside from any dynamics, which we will discuss in the next sections, it is worth discussing the dimensional constraint thoroughly.

2.2 Discussion on the dimensional constraint

Eq.(3) can be written as

$$\frac{1}{D} = \frac{1}{C} \ln \frac{a}{a_0} + \frac{1}{D_0}. \quad (4)$$

The time derivative of the above relation leads to

$$\dot{D} = -\frac{D^2}{C} \frac{\dot{a}}{a}. \quad (5)$$

In the above relation a_0 is the present scale factor of the universe corresponding to the space dimension D_0 . The values of C and δ can be calculated now in terms of other known quantities:

$$C = \frac{DD_0}{(D-D_0)} \ln\left(\frac{H_0^{-1}}{a}\right), \quad (6)$$

and

$$\log\left(\frac{\delta}{a}\right) = \frac{\log \frac{a}{H_0^{-1}}}{-1 + \frac{D}{D_0}}, \quad (7)$$

where we have replaced a_0 by the present value of the Hubble radius $H_0^{-1} = 3000h^{-1}\text{Mpc} = 9.2503 \times 10^{27}h^{-1}\text{cm}$. As the values of C and δ are not very sensitive to $0.5 < h < 1$ we take for simplicity $h = 1$.

Note that for $D \rightarrow +\infty$ the radius of the universe, $a(t)$, tends to δ (see Eq.7). In principle, δ may be greater or less than the Planck length l_{Pl} . Assuming $\delta = l_{Pl} = 1.6160 \times 10^{-33}\text{cm}$ and taking $D_0 = 3$ we obtain from (6) $C = 419.7$. Values of δ , or the minimum radius of the universe, less than the Planck length correspond to $C > 419.7$. Assuming $C < 419.7$, we obtain for the minimum radius of the universe, δ , values bigger than the Planck length. Not all dimensions of space at the present or the Planck time are of interest. We are just interested in the experimental value $D_0 = 3 - (5.3 \pm 2.5) \times 10^{-7}$ [19] based on the $g-2$ factor of electron, or $D_0 \simeq 2$ as fractal dimension for matter distribution in the universe coming from cosmological considerations [4]. At the Planck time we assume the space dimension, D_{Pl} , to be about 3 or one of the values 4, 10, or 25 coming from supergravity [2] and superstring theories [20, 21]. It is reasonable to assume that the minimum radius of the universe is less than the Planck length. Therefore, in the following we will assume $C > 419.7$. Note that, in an expanding universe, the positivity of C gives the decreasing of the spatial dimension in the course of time (cf. Eq.5). Table I shows values of C and δ for different values of D_{Pl} assuming $D_0 \simeq 3$. Similarly, table II shows the same values for $D_0 = 2$.

Figs. 1–4 demonstrate changes of C or δ vs D_{Pl} and D_0 . Figures 1–2 demonstrate the changes of C vs D_{Pl} (or $\log D_{Pl}$) for different values of D_0 in different ranges. In order to illustrate the behavior of δ vs D_{Pl} , Figs. 3 and 4 are drawn. In general, for a fixed value of D_0 , when C increases δ and D_{Pl} decrease.

TABLE I. Values of C and δ for some interesting values of D_{Pl} assuming $D_0 \simeq 3$. The reason for taking $D_0 = 3 - 5.3 \times 10^{-7}$ is based on the experimental data by measuring g -factor of electron (for details see Ref.[19]). D_{Pl} could be any value, here we take an arbitrary value for it, of course with condition $D_{Pl} \geq D_0$.

| D_0 | D_{Pl} | C | $\delta(\text{cm})$ |
|--------------------------|---------------|----------------------|------------------------------|
| $3 - 5.3 \times 10^{-7}$ | $3 + 10^{-3}$ | 1.2588×10^6 | $4.4747 \times 10^{-182210}$ |
| 3 | 4 | 1.6788×10^3 | 8.6158×10^{-216} |
| 3 | 10 | 5.9957×10^2 | 1.4771×10^{-59} |
| 3 | 25 | 4.7693×10^2 | 8.3811×10^{-42} |
| 3 | $+\infty$ | 4.1970×10^2 | 1.6160×10^{-33} |

TABLE II. Values of C and δ for different values of D_{Pl} assuming $D_0 = 2$.

| D_{Pl} | C | $\delta(\text{cm})$ |
|-----------|----------------------|--------------------------|
| 4 | 5.5960×10^2 | 2.8231×10^{-94} |
| 10 | 3.4975×10^2 | 1.0447×10^{-48} |
| 25 | 3.0413×10^2 | 8.4171×10^{-39} |
| $+\infty$ | 2.7980×10^2 | 1.6160×10^{-33} |

3 REVIEW OF THE DECRUMLING UNIVERSE MODEL

The assumption of variable space dimension brings in serious difficulties in formulating field equations. To formulate a gravitational theory based on geometry we do need a tensor calculus which is not defined for dynamical space dimension. It may be possible to write an action but we do not know how to formulate a calculus of variation in the case that the measure of the integral is itself a

dynamical variable. Therefore, we have to look for methods to define or obtain the field equations. The homogeneity and isotropy of space–time, being the main characteristics of the Friedmann models, allow us to formulate an action and a variational method to obtain the field equations. Although this do not leads to a unique model for a universe having a continuously varying dimension, it leads to interesting results and maybe the simplest way to implement the idea of having a space with variable dimension.

3.1 Action and Lagrangian

As was mentioned before, we have no way of defining an action for a general metric implementing the dynamical space dimension. Therefore, let us define it first for the special FRW metric in an arbitrary fixed space dimension D and then try to generalize it to variable dimension. Now, take the metric in constant $D + 1$ dimension in the following form (we use natural units or high energy physics units in which the fundamental constants are $\hbar = c = k_B = 1$, $G = l_{Pl}^2 = 1/m_{Pl}^2$).

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Sigma_k^2, \quad (8)$$

where $N(t)$ denotes the lapse function and $d\Sigma_k^2$ is the line element for a D –manifold of constant curvature $k = +1, 0, -1$, corresponding to the closed, flat, and hyperbolic spacelike sections, respectively. The Ricci scalar is given by

$$R = \frac{D}{N^2} \left[\frac{2\ddot{a}}{a} + (D-1) \left(\left(\frac{\dot{a}}{a} \right)^2 + \frac{N^2 k}{a^2} \right) - \frac{2\dot{a}\dot{N}}{aN} \right]. \quad (9)$$

For simplicity, let us now take $k = 0$ in our treatments as done in Refs. [11, 13]. Substituting from Eq.(9) in the Einstein–Hilbert action for pure gravity,

$$S_G = \frac{1}{2\kappa} \int d^{(1+D)}x \sqrt{-g} R, \quad (10)$$

where $\kappa = 8\pi G$, we are led to

$$\begin{aligned} S_G &= \frac{1}{2\kappa} \int d^{(1+D)}x \left\{ 2D \frac{d}{dt} \left(\frac{\dot{a}}{aN} a^D \right) - \frac{D(D-1)}{N} \left(\frac{\dot{a}}{a} \right)^2 a^D \right\} \\ &= \frac{1}{2\kappa} \int d^{(1+D)}x \left\{ \text{total time derivative} \right. \\ &\quad \left. - \frac{D(D-1)}{N} \left(\frac{\dot{a}}{a} \right)^2 a^D \right\}. \end{aligned} \quad (11)$$

The Lagrangian then becomes

$$L_G^{(0)} = - \frac{D(D-1)}{2\kappa N} \left(\frac{\dot{a}}{a} \right)^2 a^D. \quad (12)$$

Following Ref. [11], we introduce the matter Lagrangian for a perfect fluid as

$$L_M^{(0)} = - \frac{\tilde{\rho} N^2}{2} + \frac{\tilde{p} D a^2}{2}, \quad (13)$$

where

$$\tilde{\rho} := \frac{\rho}{N} a^D, \quad (14)$$

$$\tilde{p} := p N a^{D-2} \delta^{ij}, \quad (15)$$

and ρ and p being the energy density and pressure, respectively. Therefore, the complete Lagrangian is

$$L^{(0)} = - \frac{D(D-1)}{2\kappa N} \left(\frac{\dot{a}}{a} \right)^2 a^D + \left(- \frac{\tilde{\rho} N^2}{2} + \frac{\tilde{p} D a^2}{2} \right). \quad (16)$$

This Lagrangian suffers from the fact that its dimension is not $(\text{length})^{-1}$ as we know it from the $D = 3$ case, and depends on the space dimension. To remedy this shortcoming which is an obstacle for generalizing to variable dimension, we first assume that the dimension of κ is constant and equal to $(\text{length})^{D_0-1}$ for $D_0 = 3$. This is equivalent to the assumption that for κ we take the usual three dimensional gravitational constant. We then multiply the above Lagrangian by $a_0^{D_0-D}$, where a_0 is the present scale factor of the universe. For simplicity, we have assumed that a and a_0 have the dimension of length. Omitting the constant factor $a_0^{D_0}$ we arrive at

$$\mathcal{L} = -\frac{D(D-1)}{2\kappa N}(\frac{\dot{a}}{a})^2(\frac{a}{a_0})^D + (-\frac{\hat{\rho}N^2}{2} + \frac{\hat{p}Da^2}{2}), \quad (17)$$

where

$$\hat{\rho} = \frac{\rho}{N}(\frac{a}{a_0})^D, \quad (18)$$

$$\hat{p} = pa^{-2}N(\frac{a}{a_0})^D. \quad (19)$$

This Lagrangian has the dimension $(\text{length})^{-4}$ which is due to omitting of the factor $a_0^{D_0}$ for $D_0 = 3$. It is now in a form ready to generalize to any variable dimension. We consider the gravitational coupling constant κ as a constant throughout this paper. The dependence of κ on the constant spatial dimension is studied, in Ref. [22].

3.2 Field equations

For $D = D_0$, the Friedmann equations derived by the Einstein field equations are

$$\frac{1}{N^2}(\frac{\dot{a}}{a})^2 = \frac{2\kappa\rho}{D_0(D_0-1)}, \quad (20)$$

and

$$\frac{\ddot{a}}{N^2a} = \frac{\kappa}{D_0(D_0-1)}[(2-D_0)\rho - D_0p], \quad (21)$$

with $k = 0$. Note that the field equations are written in the gauge $\dot{N} = 0$. Eqs.(20, 21) are the 00- and non-vanishing ij -component of the Einstein field equations, respectively. There is an alternative Lagrangian method of deriving these field equations which can be generalized to the case of dynamical space dimension. Consider first the usual case of $D = D_0 = \text{const}$. It is easy to show that (20) is the equation of motion corresponding to variation of the Lagrangian (17) with respect to N . Moreover, variation of (17) with respect to a yields

$$-(D_0-1)[\frac{\ddot{a}}{a} + (\frac{D_0-2}{2})(\frac{\dot{a}}{a})^2] = \kappa p N^2, \quad (22)$$

which is a combination of (20 and 21). Note that in varying the Lagrangian with respect to N and a the quantities $\hat{\rho}$ and \hat{p} are considered as constants [11]–[13]. Therefore, for constant space dimensions this Lagrangian method leads to the familiar Friedman equations derived by the Einstein field equations. The continuity equation for perfect fluid is obtained from the Friedman equations (20 and 21) or (20 and 22):

$$\frac{d}{dt}(\rho a^{D_0}) + p \frac{d}{dt}(a^{D_0}) = 0. \quad (23)$$

This, in addition to an equation of state, completes the field equations for a flat FRW universe for any fixed space dimension D_0 .

To generalize the cosmological model to spaces with variable dimension, we are in principle free to choose any of the above approaches. But, it is worth noticing that the tensor calculus is meaningless in the case of dynamical dimension, since the index of a tensor is always constant and integer number. Therefore, there is no way of generalizing the Einstein equations directly. In contrast, it is possible

to write down generalized Lagrangian, such as (17), due to the fact that space is homogeneous and the dynamical part of the measure is well defined and separated. Hence, we are left with the second approach as the only possible way of formulating a cosmological model incorporating the dynamical nature of space dimension. However, we are faced now with different kind of non-uniqueness in formulating the Lagrangian or the field equations, which are discussed in the following sections.

3.2.1 Original decrumping model

Following [11], we vary the Lagrangian (17) with respect to a , assuming the gauge $\dot{N} = 0$ and taking into account the relation (3):

$$\begin{aligned} \frac{(D-1)}{N} \left\{ \frac{\ddot{a}}{a} + \left[\frac{D^2}{2D_0} - 1 - \frac{D(2D-1)}{2C(D-1)} \right] \left(\frac{\dot{a}}{a} \right)^2 \right\} \\ + N\kappa p \left(1 - \frac{D}{2C} \right) = 0. \end{aligned} \quad (24)$$

It is easily seen that this equation leads to the field equation (22) for $C \rightarrow +\infty$ and $D = D_0 = \text{const}$. Authors of [11] prefer to take the continuity equation as the second field equation. By a dimensional reasoning the continuity equation (23) is generalized to [11]

$$\frac{d}{dt} \left(\rho \left(\frac{a}{a_0} \right)^D \right) + p \frac{d}{dt} \left(\frac{a}{a_0} \right)^D = 0. \quad (25)$$

Now, a qualitative behavior of this toy model is obtained via a first integral of motion. From the Lagrangian (17), the Hamiltonian is defined as

$$\mathcal{H} = -\frac{(D-1)}{2\kappa} \frac{C^2 \dot{D}^2}{D^3} e^C \left(\frac{\delta}{a_0} \right)^D - \left(\frac{\hat{p}}{2} D \delta^2 e^{2C/D} - \frac{\hat{\rho} N^2}{2} \right), \quad (26)$$

with

$$\frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}, \quad (27)$$

and taking $\hat{\rho}$ and \hat{p} as source [1-3]:

$$\begin{aligned} \frac{d}{dt} \left[-\frac{C^2}{2\kappa N} e^C \left(\frac{\delta}{a_0} \right)^D \frac{D-1}{D^3} \dot{D}^2 - \frac{\hat{p}}{2} D \delta^2 e^{2C/D} \right] \\ = -\frac{D}{2} \delta^2 e^{2C/D} \frac{d\hat{p}}{dt}, \end{aligned} \quad (28)$$

which leads to

$$\frac{C^2}{2\kappa N} \frac{d}{dt} \left(\frac{D-1}{D^3} e^{-\frac{CD}{D_0}} \dot{D}^2 \right) + p \dot{D} e^{-\frac{CD}{D_0}} \left(\frac{1}{2} - \frac{C}{D} \right) = 0. \quad (29)$$

Taking the derivatives with respect to D instead of t , we arrive at

$$\frac{C^2}{2\kappa N} \frac{d}{dD} \left(\frac{D-1}{D^3} e^{-\frac{CD}{D_0}} \dot{D}^2 \right) + p e^{-\frac{CD}{D_0}} \left(\frac{1}{2} - \frac{C}{D} \right) = 0, \quad (30)$$

which is equivalent to

$$\begin{aligned} \frac{C^2}{2\kappa N} \left(\frac{D-1}{D^3} e^{-\frac{CD}{D_0}} \dot{D}^2 \right) + \int_{D_0}^D dD' p(D') e^{-\frac{CD'}{D_0}} \\ \times \left(\frac{1}{2} - \frac{C}{D'} \right) = 0. \end{aligned} \quad (31)$$

This is interpreted now as the equation of motion for D , having a vanishing total energy, in the potential

$$\mathcal{U}(D) := \int_{D_0}^D dD' p(D') e^{-\frac{CD'}{D_0}} \left(\frac{1}{2} - \frac{C}{D'} \right). \quad (32)$$

The kinetic energy term is given by

$$\mathcal{T} := \frac{C^2}{2\kappa N} \left(\frac{D-1}{D^3} e^{-\frac{CD}{D_0}} \dot{D}^2 \right). \quad (33)$$

The matter content is taken in the form of radiation with the following equation of state

$$p = \frac{\rho}{D}. \quad (34)$$

The kinetic energy is always positive for $D > 1$. Therefore the potential energy has to be negative for real dimensions. There is a minimum at $D = 2C$ for the potential (32) where it is negative. Its behavior at large D is given by

$$\mathcal{U}(D) \simeq \frac{D_0}{2C} e^{-C} \left(\frac{D}{D_0} \right)^{\frac{C}{D_0}}, \quad (35)$$

and for D near zero, assuming the pressure remains finite (nonzero)

$$\mathcal{U}(D) \simeq -C \ln D. \quad (36)$$

Therefore, \mathcal{U} tends to infinity when $D \rightarrow +\infty$ as well as $D \rightarrow 0$. This means that there are two turning points for D (where $\dot{D} = 0$), one above $D = 2C$ and the other below it. Appropriate assumptions about the matter content of the universe would bring the lower turning point to be $D > 1$. This leads to a model universe without any singularity. Even dimension of the model, being a dynamical quantity remains finite. The behavior of the spatial dimension with respect to the time near $D = D_0 = 3$ can be obtained by expanding (31) using $D = D_0 + \epsilon(t)$. Assuming

$$\frac{C\epsilon}{D_0} \ll 1, \quad (37)$$

they obtain to the lowest order in ϵ

$$\dot{\epsilon}^2 = \frac{2\kappa N D_0^2 p_0}{C(D_0 - 1)} \epsilon, \quad (38)$$

which has the solution

$$\epsilon = \frac{\kappa N D_0^2 p_0}{2C(D_0 - 1)} (t_0 - t)^2, \quad (39)$$

where t_0 is the time corresponding to D_0 . Using this relation, it has then been shown that for $\epsilon(t_0)$ of the order of 10^{-6} the relative change of space dimension is just about two order of magnitude within a time range of about 10 times the present accepted age of the universe [11]–[13]. That is, as we go backward in time there will be almost no change in the space dimension as far as 100 billion years.

It should be noted, however, that the change of variable from t to D in (30) is not allowed (see Sec. 3.3.2). Therefore, the result we just mentioned are questionable. Besides this, the authors of [11–13] in the course of the above calculation use the value $C = 600$, which is not consistence with assuming $D_{Pl} \simeq 3$. For this to be the case we must have $C \simeq 10^6$, as can be seen from Table I.

3.2.2 Modification of the original decrumpling model

Lima *et al.* [13] propose an alternative way of writing the field equations. They bring in the lapse function and consider the variation of the Lagrangian with respect to it as the second field equation. This corresponds to the 00-component of the Einstein field equations (cf. Eq.(20) for constant dimension):

$$\frac{1}{N^2} \left(\frac{\dot{a}}{a} \right)^2 = \frac{2\kappa\rho}{D(D-1)}. \quad (40)$$

The two field equations (24) and (40) are consistent with the continuity equation (25). Now, assuming a radiation dominated universe (i.e. $p = \rho/D$), Eqs.(5, 25, and 40) leads to the following evolution equation for the spatial dimension D

$$\dot{D}^2 = \frac{AD^3 e^{C(\frac{D}{D_0}-1)} (\frac{D}{D_0})^{\frac{C}{D_0}}}{C^2(D-1)}, \quad (41)$$

where $A = 2\kappa\rho_0 N^2$ and ρ_0 is the energy density of the universe corresponding to $D = D_0$. This “energy equation” for D corresponds now to (31). However, there is an inconsistency between them (see Sec. 3.3.2). For $D < 1$ the kinetic term is again negative. Now, the potential energy is always negative for $1 < D < +\infty$. Thus there is no turning point in this modified model.

3.3 Critique of the original decrumpling Model and its modification

As explained before, there is not a unique formulation for the cosmological models with variable space dimension, and no principle from which we could derive unique field equations. Therefore, the observational consequences are the only way of looking for validity of the models. Despite this non-uniqueness and possible observational viability, we would like to mention some theoretical deficiencies of the above models.

3.3.1 Measure of Action

The main difficulty in formulating a variational calculus for a theory with variable space dimension is the dependence of the measure of action on dimension, making the measure a dynamical quantity. As we will see in the next section, this dependence seriously influences the field equations. Models discussed up to now [11, 13] do not consider this dependence. Taking into account the measure of the integral will also change the equation of continuity (see Eq.69).

3.3.2 Incorrect time evolution equation of the spatial dimension

We have seen that two equations (31) and (41) leads to quite different results regarding the behavior of D . Lima *et al.* [13] are able to use actually one of the field equations directly, i.e. (40). Authors of Ref. [11], however, use equation (31), which is based on a non-valid change of variables in going from Eq.(29) to (30). Therefore, their equation for the dynamics of D , i.e. (31), is not valid.

The point is better understood if we write the Hamiltonian in its correct canonical form. Substituting the momentum conjugate to the space dimension, p_D , in (26), we obtain for the Hamiltonian

$$\begin{aligned} \mathcal{H}(p_D, D) = & -\frac{\kappa N D^3 e^{-C(1-\frac{D}{D_0})}}{2C^2(D-1)} p_D^2 + \frac{1}{2}(\hat{\rho} N^2 \\ & - \hat{p} D \delta^2 e^{2C/D}). \end{aligned} \quad (42)$$

Now, using \mathcal{H} as a first integral of motion (see Eq.27), we obtain

$$\frac{\kappa N}{2C^2} \frac{d}{dt} \left(\frac{D^3 e^{C D/D_0}}{D-1} p_D^2 \right) + p e^{C(2-\frac{D}{D_0})} \dot{D} \left(\frac{1}{2} - \frac{C}{D} \right) = 0. \quad (43)$$

Since p_D and D are independent variables, it is obvious that we are not allowed to change the differential variable from t to D . Note that Eq.(43) is just Eq.(29) where p_D is substituted in terms of \dot{D} . Indeed, it is easy to show that Eq.(29) or (43) leads to (24).

4 NEW MODELS FOR A UNIVERSE WITH VARIABLE SPACE DIMENSION

We have mentioned some of the shortcomings of the original model, regarding the field equations and its results. Now, as we have mentioned before even the Lagrangian is not unique. Here we try to

remedy some of the shortcomings and also write down other possible Lagrangians.

The matter part of the Lagrangian was written in a form such that $\hat{\rho}$ and \hat{p} could be taken as constant in the variation of Lagrangian. Now, for the case of variable dimension this procedure may not work. Therefore, we prefer to use another procedure to write down the matter Lagrangian which is due to Hawking and Ellis [17]. This is outlined in section 4.1. In section 4.2, we formulate new actions, and Lagrangians, and then derive the corresponding field equations, using the new matter Lagrangian (54) and implementing the dynamic character of the measure.

4.1 Hawking–Ellis action of a perfect fluid in constant space dimension

We want to obtain the energy–momentum tensor of a perfect fluid in constant dimension, via an action formalism, in the form

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu}, \quad (44)$$

where U is the timelike velocity four vector satisfying

$$g_{\alpha\beta}U^\alpha U^\beta = -1. \quad (45)$$

The fluid can be described by a function μ , called the density, and a congruence of timelike flow lines. The fluid current vector, defined by $j^\alpha = \mu U^\alpha$, is conserved:

$$j^\alpha{}_{;\alpha} = 0, \quad (46)$$

where “ $;$ ” denotes covariant derivative. Taking the elastic potential (or internal energy) ϵ as a function of μ the action is written as [17]

$$S_M = - \int d^{(D+1)}x \sqrt{-g} \mu (1 + \epsilon). \quad (47)$$

The calculations may be simplified by noting that the conservation of the current can be expressed as

$$j^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} j^\alpha) = 0. \quad (48)$$

Given the flow lines, the conservation equations determine j^α uniquely at each point on a flow line in terms of the initial values at some given point on the same flow line. Therefore, $(\sqrt{-g})j^\alpha$ is unchanged when the metric is varied. But

$$\mu^2 = -g^{-1}((\sqrt{-g}j^\alpha)(\sqrt{-g}j^\beta))g_{\alpha\beta}, \quad (49)$$

so

$$2\mu\delta\mu = (j^\alpha j_\alpha g^{\mu\nu} - j^\mu j^\nu)\delta g_{\mu\nu}. \quad (50)$$

Now, using the definition of energy–momentum tensor [23] as

$$\delta S_M = \frac{1}{2} \int d^{(D+1)}x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}, \quad (51)$$

one gets Eq. (44) for the energy momentum–tensor of a perfect fluid in which

$$\rho = \mu(1 + \epsilon), \quad (52)$$

and

$$p = \mu^2 \left(\frac{d\epsilon}{d\mu} \right), \quad (53)$$

are the energy density and the pressure respectively. Hence the action for a perfect fluid may be expressed as

$$S_M = - \int d^{(D+1)}x \sqrt{-g} \rho. \quad (54)$$

Taking a FRW metric given by Eq.(8), it is easy to see that for a comoving observer whose contravariant velocity four vector is

$$U^\alpha = (N^{-1}, 0, \dots, 0), \quad (55)$$

the energy-momentum tensor is given by

$$T^{\mu\nu} = \text{diag}(\rho N^{-2}, pa^{-2}, \dots, pa^{-2}). \quad (56)$$

In the variation of S_M to obtain the energy-momentum tensor we have used the following relations:

$$\delta\mu = -\frac{\mu D}{a}\delta a, \quad \text{if } a \rightarrow a + \delta a, \quad (57)$$

and

$$\delta\mu = 0, \quad \text{if } N \rightarrow N + \delta N. \quad (58)$$

The matter action (54) is now in suitable form to be generalized to variable space dimension.

4.2 Definition of our model universe and its field equations

We are now in a position to write down the complete action for matter and gravity in D space dimension. Here our treatments are for each kind of D -dimensional topology (closed, flat, and open), in contrast to the earlier works in [11–13] which are for the special case of a flat FRW universe. Using the relations (9, 10, 54) we obtain

$$\begin{aligned} S_0 &:= S_G + S_M \\ &= \int d^{(1+D)}x N a^D \left\{ \frac{D}{2\kappa N^2} \left[\frac{2\ddot{a}}{a} + (D-1) \left(\frac{\dot{a}}{a} \right)^2 \right. \right. \\ &\quad \left. \left. + \frac{N^2 k}{a^2} \right) - \frac{2\dot{a}\dot{N}}{aN} \right] - \rho \right\}. \end{aligned} \quad (59)$$

Now, trying again to make the action dimensionless, we have to multiply it by the factor $a_0^{D_0-D}$ (cf. Sec. 3.1). Omitting the constant part of this factor, $a_0^{D_0}$, and integrating over the space part of the action (59), we reach to

$$\begin{aligned} S &:= S_0 a^{-D_0} \\ &= \int dt V_D N \left(\frac{a}{a_0} \right)^D \left\{ \frac{D}{2\kappa N^2} \left[\frac{2\ddot{a}}{a} + (D-1) \left(\frac{\dot{a}}{a} \right)^2 \right. \right. \\ &\quad \left. \left. + \frac{N^2 k}{a^2} \right) - \frac{2\dot{a}\dot{N}}{aN} \right] - \rho \right\}, \end{aligned} \quad (60)$$

where V_D is the volume of the spacelike sections coming from the space integration due to the homogeneity and the isotropy of the FRW metric. Assuming D to be an integer, V_D is given by

$$V_D = \begin{cases} \frac{2\pi^{(\frac{D+1}{2})}}{\Gamma(\frac{D+1}{2})}, & \text{if } k = +1, \\ \frac{\pi^{(\frac{D}{2})}}{\Gamma(\frac{D}{2}+1)} \chi_c^D, & \text{if } k = 0, \\ \frac{2\pi^{(\frac{D}{2})}}{\Gamma(\frac{D}{2})} f(\chi_c), & \text{if } k = -1, \end{cases} \quad (61)$$

where χ_c is a cut-off and $f(\chi_c)$ is a function thereof (see Appendix A for more details). Now let us assume D to be a dynamical parameter accepting any non-integer value. So, the volume of spacelike sections, V_D , must be defined for fractal or non-integer dimensions. Since the volume of a fractal structure having non-integer dimension is beyond the scope of this paper, we take the expression (61) to be valid for the volume of spacelike sections whether the spatial dimension is integer or not.

Now, the action (60) depends on the second time derivative of the dynamical variable a . We can get

rid of it through introduction of a total time derivative, as it was done in section 3.1. We, however, intend to generalize our action to the case of variable space dimension. Therefore we have to decide which operation being done first: generalization to variable space dimension or substitution of the second derivative with a total time derivative, as it was done in the case of the original decrumpling model in section 3.1 [13]. Substituting first the term \ddot{a} with a total time derivative we obtain

$$S_I := \frac{1}{2\kappa} \int dt \left\{ \frac{2DV_D}{a_0^D} \frac{d}{dt} \left(\frac{\dot{a}a^D}{aN} \right) - \frac{V_D D(D-1)}{N} \left(\frac{\dot{a}}{a} \right)^2 - \frac{N^2 k}{a^2} \left(\frac{a}{a_0} \right)^D - \rho N V_D \left(\frac{a}{a_0} \right)^D \right\}. \quad (62)$$

Omitting the total time derivative in the action (62), we reach at the following Lagrangian

$$L_I := -\frac{1}{2\kappa} \left(\frac{a}{a_0} \right)^D \frac{D(D-1)}{N} \left(\frac{\dot{a}}{a} \right)^2 - \frac{N^2 k}{a^2} \left(\frac{a}{a_0} \right)^D - \rho N V_D \left(\frac{a}{a_0} \right)^D. \quad (63)$$

In the case $k = 0$, this is the same Lagrangian as (17) except for introducing the dimensional dependence of the volume and the special form of the matter Lagrangian. As we are going to work in the gauge $\dot{N} = 0$ we could derive first the field equations and then insert the gauge condition, or assume first that $N = \text{constant}$ and then vary the Lagrangian. But it is easily seen that both procedures lead to the same result. Therefore, we will assume from now on, without loss of generality, that $N = \text{constant}$.

Now, we assume the space dimension D to be a variable and use the constraint (3) in order to reduce the dynamical variables to N and a . The equations of motion are then obtained by variation of the Lagrangian (63) with respect to N and a . For the variation of ρ we do need the following relations based on Eqs.(48, 49) and (50):

$$j^\alpha{}_{;\alpha} = \frac{1}{NV_D \left(\frac{a}{a_0} \right)^D} \frac{\partial}{\partial x^\alpha} (NV_D \left(\frac{a}{a_0} \right)^D j^\alpha) = 0, \quad (64)$$

$$\mu^2 = -\frac{1}{N^2 V_D^2 \left(\frac{a}{a_0} \right)^{2D}} [(NV_D \left(\frac{a}{a_0} \right)^D j^\alpha) (NV_D \left(\frac{a}{a_0} \right)^D j^\beta)] g_{\alpha\beta}, \quad (65)$$

and

$$\delta\mu = -\mu \left[\left(\frac{d \ln V_D}{dD} + \ln \frac{a}{a_0} \right) \frac{dD}{da} + \frac{D}{a} \right] \delta a. \quad (66)$$

As mentioned in (58) variation of μ with respect to N is zero for constant dimension. It is easy to see that this fact is still true in the case of variable dimension.

Now, using (66) and the definition of the energy density and the pressure, as given by Eqs.(52) and (53) the following equations of motion are obtained:

$$\frac{1}{N^2} \left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{2\kappa\rho}{D(D-1)}, \quad (67)$$

and

$$\begin{aligned} (D-1) \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{N^2 k}{a^2} \right] & - \frac{D^2}{2C} \frac{d \ln V_D}{dD} \\ & - 1 - \frac{D(2D-1)}{2C(D-1)} + \frac{D^2}{2D_0} + \kappa p N^2 \left(-\frac{d \ln V_D}{dD} \frac{D}{C} \right. \\ & \left. - \frac{D}{C} \ln \frac{a}{a_0} + 1 \right) = 0. \end{aligned} \quad (68)$$

Using Eqs.(67,68), one easily gets the continuity equation for our model with variable space dimension:

$$\frac{d}{dt}(\rho(\frac{a}{a_0})^D V_D) + p \frac{d}{dt}((\frac{a}{a_0})^D V_D) = 0. \quad (69)$$

This continuity equation can be integrated for the case of dust or radiation to obtain the energy density as a function of time. For radiation era, $p = \frac{\rho}{D}$, we have

$$\rho = \rho_0 e^{C(\frac{D}{D_0}-1)} (\frac{D}{D_0})^{\frac{C}{D_0}} \frac{V_{D_0}}{V_D} e^{-\int_{D_0}^D dD \frac{1}{D} \frac{d \ln V_D}{dD}}, \quad (70)$$

and for matter era, $p = 0$:

$$\rho = \rho_0 e^{C(\frac{D}{D_0}-1)} \frac{V_{D_0}}{V_D}. \quad (71)$$

Using Eqs.(5, 67), the evolution equation for the spatial dimension is derived as

$$\dot{D}^2 = \frac{N^2 D^4}{C^2} [\frac{2\kappa\rho}{D(D-1)} - k\delta^{-2} e^{-2C/D}]. \quad (72)$$

However, if we assume the time variability of D first, substitution of the second time derivative by a total time derivative leads to the following action

$$\begin{aligned} S_{II} := & \frac{1}{2\kappa} \int dt \{ 2 \frac{d}{dt} (\frac{DV_D}{N} \frac{\dot{a}}{a} (\frac{a}{a_0})^D) - \frac{2\dot{D}\dot{a}V_D}{aN} (\frac{a}{a_0})^D \\ & - 2 \frac{DV_D}{N} \frac{\dot{a}}{a} \ln \frac{a}{a_0} (\frac{a}{a_0})^D - \frac{2D\dot{D}}{N} \frac{\dot{a}}{a} (\frac{a}{a_0})^D \frac{dV_D}{dD} \\ & - \frac{V_D D(D-1)}{N} ((\frac{\dot{a}}{a})^2 - \frac{N^2 k}{a^2}) (\frac{a}{a_0})^D \\ & - \rho N V_D (\frac{a}{a_0})^D \}. \end{aligned} \quad (73)$$

Neglecting now the total time derivative, we reach at the following Lagrangian

$$\begin{aligned} L_{II} := & -\frac{V_D}{2\kappa} (\frac{a}{a_0})^D \{ \frac{2\dot{D}\dot{a}}{aN} + \frac{2D\dot{a}\dot{D}}{aN} \ln \frac{a}{a_0} + \frac{D(D-1)}{N} \\ & \times ((\frac{\dot{a}}{a})^2 - \frac{N^2 k}{a^2}) + \frac{2D\dot{D}}{N} \frac{\dot{a}}{a} (\frac{a}{a_0})^D \frac{d \ln V_D}{dD} \} \\ & - \rho V_D N (\frac{a}{a_0})^D. \end{aligned} \quad (74)$$

This Lagrangian is more general than L_I . Putting $\dot{D} = 0$ in L_{II} we obtain the Lagrangian L_I which can still be considered as a Lagrangian with variable space dimension.

Using again the dimensional constraint (3) and Eqs.(52, 53) and (66), variation of L_{II} with respect to N and a leads to the following equations of motion

$$\frac{1}{N^2} (\frac{\dot{a}}{a})^2 = \frac{2\kappa\rho - \frac{kD(D-1)}{a^2}}{D[\frac{2D^2}{D_0} - D - 1 - \frac{2D}{C} - \frac{2D^2}{C} \frac{d \ln V_D}{dD}]}, \quad (75)$$

and

$$\begin{aligned} \frac{\ddot{a}}{a} [\frac{2D}{C} + \frac{2D^2}{C} \ln \frac{a}{a_0} + \frac{2D^2}{C} \frac{d \ln V_D}{dD} - D + 1] \\ + (\frac{\dot{a}}{a})^2 [-\frac{D^4}{C^2} (\ln \frac{a}{a_0})^2 + (-\frac{4D^3}{C^2} + \frac{3D^3}{2C} - \frac{5D^2}{2C} \ln \frac{a}{a_0} \end{aligned}$$

$$\begin{aligned}
& -\frac{d \ln V_D}{dD} \left(\frac{4D^3}{C^2} - \frac{3D^3}{2C} + \frac{5D^2}{2C} + \frac{2D^4}{C^2} \ln \frac{a}{a_0} \right) \\
& - \frac{D^4}{C^2 V_D} \frac{d^2 V_D}{dD^2} + \frac{3D^2}{C} - \frac{2D^2}{C^2} - \frac{5D}{2C} - \frac{(D-1)(D-2)}{2} \Big] \\
& + \frac{kN^2}{a^2} \left[\frac{D(2D-1)}{2C} + \frac{D^2(D-1)}{2C} \frac{d \ln V_D}{dD} \right. \\
& \left. - \frac{(D-1)(D-2)}{2} + \frac{D^2(D-1)}{2C} \ln \frac{a}{a_0} \right] + N^2 \kappa p \left(\frac{D}{C} \ln \frac{a}{a_0} \right. \\
& \left. + \frac{D}{C} \frac{d \ln V_D}{dD} - 1 \right) = 0.
\end{aligned} \tag{76}$$

In the limit $C \rightarrow +\infty$ and $D = D_0 = \text{const}$, Eqs.(67, 68, 75, 76) reduce to the standard Friedmann equations (20,22). Using Eqs.(75, 76), one easily gets the continuity equation for our model with variable space dimension. Now, using Eqs.(5, 75), the evolution equation for the spatial dimension is derived as

$$\dot{D}^2 = \frac{N^2 D^3 [2\kappa\rho - kD(D-1)\delta^{-2}e^{-\frac{2C}{D}}]}{C^2 [\frac{2D^2}{D_0} - D - 1 - \frac{2D}{C} - \frac{2D^2}{C} \frac{d \ln V_D}{dD}]} \tag{77}$$

From Eqs.(72) and (77), it is easy to see that in the limit $C \rightarrow +\infty$, we have $\dot{D} = 0$, which corresponds to the standard universe model with constant dimension. To understand the dynamical behavior of dimension in the models discussed we have to substitute ρ and V_D from (70–71) and (61) in (72) or (77) respectively. In the case of $k = 0$ it is, however, seen easily that there is no turning point in dimension and both models can be contracted up to $a = \delta$ corresponding to $D \rightarrow +\infty$. The discussion can be simplified for a de Sitter–like model which we will discuss in the next section.

4.3 de Sitter–like model

Consider a universe dominated by the constant vacuum energy, ρ_Λ , corresponding to a cosmological constant Λ and the equation of state $p_\Lambda = -\rho_\Lambda$. Note that these relations are satisfied by the continuity equation (69). Substituting

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}, \tag{78}$$

in (72) and (77), we obtain evolution equation of the spatial dimension for the Lagrangian L_I

$$\dot{D}^2 = \frac{-N^2 D^3}{C^2 (D-1)} [D(D-1)\delta^{-2}e^{-2C/D} - 2\Lambda] \equiv -V_I(D), \tag{79}$$

and for the Lagrangian L_{II}

$$\dot{D}^2 = -\frac{N^2 D^3 [D(D-1)\delta^{-2}e^{-2C/D} - 2\Lambda]}{C^2 [\frac{2D^2}{D_0} - \frac{2D^2}{C} \frac{d \ln V_D}{dD} - \frac{2D}{C} - D - 1]} \equiv -V_{II}(D), \tag{80}$$

where we have set $k = +1$. As was mentioned in Sec. 3.2, these equations may be interpreted as “energy equation” for D , having vanishing total energy with the potentials $V_I(D)$ and $V_{II}(D)$ respectively. Both models have a “classical” turning point at D_T , where \dot{D} vanishes. This is the point of maximum dimension of space corresponding to the minimum radius of the universe a_T (see Fig. 5). We obtain this dimension as the solution of the equation

$$D(D-1)\delta^{-2}e^{-2C/D} - 2\Lambda = 0. \tag{81}$$

In what follows we will choose for the cosmological constant

$$\Lambda = 3G^{-1} = 3l_{Pl}^{-2}. \tag{82}$$

Eq.(81) has to be solved numerically. Table III shows some interesting D_T values. There is a minimum dimension for both models which is reached asymptotically. This minimum dimension for model I is

$D = 1$ corresponding to a maximum radius $a_{Im} \equiv \delta e^C$. To study the asymptotic behavior of $V_{II}(D)$, we need to calculate the term $\frac{d \ln V_D}{dD}$. From (61), for $k = +1$, we obtain

$$\frac{d \ln V_D}{dD} = \frac{1}{2}(\ln \pi - \psi(\frac{D+1}{2})), \quad (83)$$

where ψ is the logarithmic derivative of the gamma function, $\psi \equiv \frac{\Gamma'(x)}{\Gamma(x)}$, called Euler's Psi function. Substituting (83) into (80), it is easy to see that the potential $V_{II}(D)$ has an asymptotic behavior at $D_{II m}$, corresponding to $a_{II m}$, being the solution of the equation

$$\frac{2D^2}{D_0} - \frac{D^2}{C}(\ln \pi - \psi(\frac{D+1}{2})) - \frac{2D}{C} - D - 1 = 0. \quad (84)$$

Some numerical values for $D_{II m}$ and $a_{II m}$ are also given in Table III. A generic example of the potential $V_{II}(D)$ is plotted in Fig. 5. It exhibits the classically allowed region ($\dot{D}^2 > 0$). Therefore in a de Sitter-like universe, taking the model II, for the dimension of space we have $D_{II m} < D < D_T$. D_T , which corresponds to the minimum contraction in the de Sitter-like model, can have very different dimensions, as given in Table III. There we can also see that the minimum scale factor, a_T , corresponding to D_T can be up to about 10 times the Planck length, assuming $D_0 = 3$. The assumption $D_0 = 2$ leads to almost the same result with values for D_T a bit smaller than those of Table III. The maximum radius $a_{II m}$, corresponding to the minimum dimension $D_{II m}$ which is about 2 as can be seen from Table III, is many order of magnitudes bigger than the present radius of the universe.

The behavior of $V_I(D)$ is qualitatively the same as $V_{II}(D)$, except that the minimum dimension is at $D_{II m} \equiv 1$. In the next sections on quantum cosmology of our de Sitter-like models we will come back to the details of this behavior. The detailed behavior of the potentials and the field equations leads to interesting cosmological consequences which we will discuss in a forthcoming paper [24].

TABLE III. a_T/l_{Pl} , D_T , $\log(a_{II m}/l_{Pl})$, $D_{II m}$, $\log(a_{Im}/l_{Pl})$ for $k = +1$ and for interesting values of C corresponding to $D_{Pl} = 3.001, 4, 10, 25$ and $D_0 \simeq 3$ (cf Table I for corresponding values of δ).

| C | a_T/l_{Pl} | D_T | $\log(a_{II m}/l_{Pl})$ | $D_{II m}$ | $\log(a_{Im}/l_{Pl})$ |
|----------------------|--------------|-------|-------------------------|------------|-----------------------|
| 1.2588×10^6 | 1.0 | 3.001 | 7.8160×10^4 | 2.0 | 3.6452×10^5 |
| 1.6788×10^3 | 1.4 | 3.996 | 1.5079×10^2 | 2.189 | 5.4682×10^2 |
| 5.9957×10^2 | 3.0 | 9.782 | 9.2644×10 | 2.194 | 2.3435×10^2 |
| 4.7693×10^2 | 8.0 | 22.42 | 8.6034×10 | 2.196 | 1.9884×10^2 |

5 WAVE FUNCTION OF OUR MODEL UNIVERSE

We are now interested in quantum cosmological behavior of our model universe and its differences to the Wheeler-DeWitt equation and its solutions for the de Sitter minisuperspace in 3-space. We will therefore use the canonical approach of quantization due to DeWitt and Wheeler [25, 26]. To do so, let us first briefly review the canonical quantization of the universe with constant spatial dimension.

5.1 Brief review of the tunneling and the Hartle-Hawking wave function

According to the quantum approach to the standard cosmology [27, 28, 29], a small closed universe can spontaneously nucleate out of “nothing”, where by “nothing” we mean a state with no classical space-time. The cosmological wave function can be used to calculate the probability distribution for the initial configuration of the nucleating universe. Once the universe nucleated, it is expected to go through a period of inflation.

Let us now illustrate how the nucleation of the universe can be described in the simplest de Sitter minisuperspace model. In this model, the universe is assumed to be homogeneous, isotropic, closed, and filled with a vacuum of constant energy density ρ_Λ . The universe should be closed, since otherwise its volume will be infinite and the nucleation probability would be zero. The radius of the universe a

is the only dynamical variable of the model, and the wave function $\Psi(a)$ satisfies the Wheeler–DeWitt (WDW) equation:

$$H\Psi(a) = 0, \quad (85)$$

where H is the corresponding Hamiltonian of the model. In our de Sitter case it is written as

$$\left[\frac{d^2}{da^2} - \frac{9\pi^2 a^2}{4G^2}(1 - l_\Lambda^{-2}a^2)\right]\Psi(a) = 0. \quad (86)$$

Here

$$l_\Lambda^{-2} \equiv \frac{8\pi G\rho_\Lambda}{3} = \frac{\Lambda}{3}. \quad (87)$$

The value $\Lambda = 3G^{-1}$ corresponds to $l_\Lambda = l_{Pl}$. We have disregarded the ambiguity in the ordering of non-commuting operators a and d/da , which is unimportant in the semiclassical approximation we are using here. Eq.(86) has the form of a one-dimensional Schrödinger equation for a “particle” described by a coordinate $a(t)$ having zero energy and moving in the potential

$$U(a) = \frac{9\pi^2 a^2}{4G^2}(1 - l_\Lambda^{-2}a^2). \quad (88)$$

The classically allowed region is $a \geq l_\Lambda$, and the WKB solution of Eq.(86) in this region is

$$\Psi_\pm(a) = [p(a)]^{-1/2} \exp[\pm i \int_{l_\Lambda}^a p(a') da' \mp i\pi/4], \quad (89)$$

where $p(a) = [-U(a)]^{1/2}$. The under-barrier ($a < l_\Lambda$) solutions are

$$\tilde{\Psi}_\pm(a) = |p(a)|^{-1/2} \exp[\pm \int_a^{l_\Lambda} |p(a')| da']. \quad (90)$$

The classical momentum conjugate to a is $p_a = -\frac{3\pi a \dot{a}}{2GN}$. For $a \gg l_\Lambda$ we have

$$\hat{p}_a \Psi_\pm(a) \approx \pm p(a) \Psi_\pm(a), \quad (91)$$

where $\hat{p}_a = -i\partial/\partial a$. Thus $\Psi_-(a)$ and $\Psi_+(a)$ describe an expanding and a contracting universe, respectively. According to Vilenkin, the tunneling boundary condition [30] requires that only the expanding component should be present at large a . Therefore we obtain for the Vilenkin tunneling wave function Ψ_V

$$\Psi_V(a > l_\Lambda) = \Psi_-(a). \quad (92)$$

The under-barrier wave function is then found from WKB connection formula:

$$\Psi_V(a < l_\Lambda) = \tilde{\Psi}_+(a) - \frac{i}{2} \tilde{\Psi}_-(a). \quad (93)$$

The growing exponential $\tilde{\Psi}_-(a)$ and the decreasing exponential $\tilde{\Psi}_+(a)$ have comparable amplitudes at the nucleation point $a = l_\Lambda$, but away from that point the decreasing exponential dominates (see Fig. 6). Therefore, the nucleation probability can be approximated as [18]

$$\mathcal{P} \sim \left|\frac{\Psi_V(l_\Lambda)}{\Psi_V(0)}\right|^2 \sim \exp[-2 \int_0^{l_\Lambda} |p(a')| da'] = \exp(-\frac{3\pi}{G\Lambda}). \quad (94)$$

The Hartle–Hawking wave function satisfying the no-boundary boundary condition takes the form [28]

$$\Psi_{HH}(a < l_\Lambda) = \tilde{\Psi}_-(a), \quad (95)$$

for the under-barrier wave function and

$$\Psi_{HH}(a > l_\Lambda) = \Psi_+(a) + \Psi_-(a), \quad (96)$$

in the classically allowed range. This wave function describes a contracting and re-expanding universe; under the barrier $\Psi_{HH}(a)$ is exponentially suppressed (see Fig. 6).

Using the anti-Wick rotation for Euclideanization of a Lorentzian path integral, Linde [29] suggested his tunneling wave function. Linde's wave function has only the decreasing exponential $\tilde{\Psi}_+(a)$ in the non-classically allowed region:

$$\Psi_L(a < l_\Lambda) = \tilde{\Psi}_+(a). \quad (97)$$

The continuation to the classically allowed region gives

$$\Psi_L(a > l_\Lambda) = \frac{1}{2}[\Psi_+(a) + \Psi_-(a)]. \quad (98)$$

Different probability densities are assumed to correspond to different boundary conditions implied by the Hartle–Hawking “no-boundary” proposal [28], the “tunneling” proposal of Vilenkin [30, 31], and that of Linde [29]. In particular, the nucleation probability for instanton-dominated transitions is assumed to be

$$\mathcal{P} \propto |\Psi|^2 \propto \begin{cases} \exp(-2S_E) & \text{for } \Psi_{HH}, \\ \exp(2S_E) & \text{for } \Psi_L, \Psi_V, \end{cases} \quad (99)$$

where S_E is the Euclidean action of the instanton. Problems concerning the interpretation of the wave function of the universe are studied in many articles [33]. For a recent discussion of problems associated with defining the initial cosmological wave function see e.g. Ref.[34]. The debate about the form of the wave function of the universe has recently intensified by Vilenkin [35], Hawking and Turok [36], and Linde [37]; see also Refs.[38, 39].

5.2 Wheeler–DeWitt equation in the universe with dynamical spatial dimension

Turning now to the canonical quantization of our model with variable space dimension, we consider the simplest case of the de Sitter minisuperspace. We therefore take $k = +1$ and $\rho = \rho_\Lambda = \frac{\Lambda}{8\pi G}$, in the Lagrangian L_I and L_{II} . The corresponding Hamiltonians to L_I and L_{II} can be written in the form

$$\begin{aligned} H_I &:= N\mathcal{H}_I \\ &:= p_{Ia}\dot{a} - L_I \\ &= N\left\{-\frac{\kappa a^2 p_{Ia}^2}{2V_D(\frac{a}{a_0})^D D(D-1)} - \frac{V_D D(D-1)}{2\kappa a^2}(\frac{a}{a_0})^D\right. \\ &\quad \left.+ \frac{V_D \Lambda}{\kappa}(\frac{a}{a_0})^D\right\}, \end{aligned} \quad (100)$$

and

$$\begin{aligned} H_{II} &:= N\mathcal{H}_{II} := p_{IIa}\dot{a} - L_{II} \\ &= N\left\{-\frac{\kappa a^2 p_{IIa}^2}{4DV_D(\frac{a}{a_0})^D [\frac{D^2}{D_0} - \frac{D^2}{C} \frac{d \ln V_D}{dD} - \frac{D}{C} - \frac{D+1}{2}]}\right. \\ &\quad \left.- \frac{D(D-1)V_D}{2\kappa a^2}(\frac{a}{a_0})^D + \frac{V_D \Lambda}{\kappa}(\frac{a}{a_0})^D\right\}, \end{aligned} \quad (101)$$

where V_D is the volume of D-sphere S^D (cf. Appendix A and Eq.61). The canonical momenta p_{Ia} and p_{IIa} are defined as

$$\begin{aligned} p_{Ia} &:= \frac{\partial L_I}{\partial \dot{a}} \\ &= -\frac{V_D D(D-1)\dot{a}}{\kappa N a^2}(\frac{a}{a_0})^D, \end{aligned} \quad (102)$$

and

$$\begin{aligned}
p_{IIa} &:= \frac{\partial L_{II}}{\partial \dot{a}} + \frac{\partial L_{II}}{\partial \dot{D}} \frac{\partial \dot{D}}{\partial \dot{a}} \\
&= -\frac{V_D D \dot{a}}{\kappa N a^2} \left(\frac{a}{a_0}\right)^D \left[\frac{2D^2}{D_0} - \frac{2D^2}{C} \frac{d \ln V_D}{dD} \right. \\
&\quad \left. - \frac{2D}{C} - D - 1 \right].
\end{aligned} \tag{103}$$

The above systems can be quantized by the assignment

$$p_{Ia}^2, p_{IIa}^2 \rightarrow -a_0^{-2D_0} \frac{1}{a^q} \frac{\partial}{\partial a} a^q \frac{\partial}{\partial a}, \tag{104}$$

where q is the factor ordering parameter. Note that we have introduced the coefficient $a_0^{-2D_0}$ in (104) for the kinetic term in WDW equation to have the right dimension. The WDW equation is obtained by applying the classical constraint

$$\frac{\delta H_I}{\delta N} = 0, \tag{105}$$

$$\frac{\delta H_{II}}{\delta N} = 0, \tag{106}$$

on the wave function $\Psi_I(a)$ and $\Psi_{II}(a)$ respectively:

$$\begin{aligned}
\mathcal{H}_I \Psi_I(a) &= \left\{ \frac{1}{a^q} \frac{\partial}{\partial a} a^q \frac{\partial}{\partial a} - \frac{a_0^{2D_0} V_D^2 D(D-1)}{\kappa^2 a^2} \left(\frac{a}{a_0}\right)^{2D} \right. \\
&\quad \times \left. \left(\frac{D(D-1)}{a^2} - 2\Lambda \right) \right\} \Psi_I(a) = 0,
\end{aligned} \tag{107}$$

and

$$\begin{aligned}
\mathcal{H}_{II} \Psi_{II}(a) &= \left\{ \frac{1}{a^q} \frac{\partial}{\partial a} a^q \frac{\partial}{\partial a} - \frac{a_0^{2D_0} V_D^2 D}{\kappa^2 a^2} \left(\frac{a}{a_0}\right)^{2D} \right. \\
&\quad \times \left. \left(\frac{D(D-1)}{a^2} - 2\Lambda \right) \left[\frac{2D^2}{D_0} - \frac{2D^2}{C} \frac{d \ln V_D}{dD} \right. \right. \\
&\quad \left. \left. - \frac{2D}{C} - D - 1 \right] \right\} \Psi_{II}(a) = 0.
\end{aligned} \tag{108}$$

In the semiclassical approximation the ambiguity in the ordering of a and d/da can be ignored. Therefore, we will assume $q = 0$. The corresponding potentials for $\Psi_I(a)$ and $\Psi_{II}(a)$ are

$$\begin{aligned}
U_I(a) &:= \frac{a_0^{2D_0} V_D^2 D(D-1)}{\kappa^2 a^2} \left(\frac{a}{a_0}\right)^{2D} \\
&\quad \times \left(\frac{D(D-1)}{a^2} - 2\Lambda \right),
\end{aligned} \tag{109}$$

$$\begin{aligned}
U_{II}(a) &:= \frac{a_0^{2D_0} V_D^2 D}{\kappa^2 a^2} \left(\frac{a}{a_0}\right)^{2D} \left(\frac{D(D-1)}{a^2} - 2\Lambda \right) \left[\frac{2D^2}{D_0} \right. \\
&\quad \left. - \frac{2D^2}{C} \frac{d \ln V_D}{dD} - \frac{2D}{C} - D - 1 \right].
\end{aligned} \tag{110}$$

It is worth noticing that in these equations D is a function of the scale factor, a , according to the constraint (3). It can be easily shown

$$U_I(a) \rightarrow U(a), \quad \text{as } D = D_0 = 3,$$

and

$$U_{II}(a) \rightarrow U(a), \quad \text{as } D = D_0 = 3, \quad C \rightarrow \infty.$$

Note that C appears explicitly in $U_{II}(a)$. This is due to appearance of the time derivative of D in the Lagrangian L_{II} . The shape of the potential $U_I(a)$ and $U_{II}(a)$ determine how we impose the appropriate boundary condition and evaluate the wave function of our model. Using (61) for $k = +1$ and (83) and the dimensional constraint (3), the potentials $U_I(a)$ and $U_{II}(a)$ can be written in terms of D only:

$$\begin{aligned} U_I(D) &= \frac{4a_0^{2D_0} \pi^{(D+1)} D(D-1) \delta^{-2} e^{2C(1-\frac{D}{D_0}-\frac{1}{D})}}{\kappa^2 (\Gamma(\frac{D+1}{2}))^2} \\ &\times (D(D-1) \delta^{-2} e^{-2C/D} - 2\Lambda), \end{aligned} \quad (111)$$

and

$$\begin{aligned} U_{II}(D) &= \frac{4a_0^{2D_0} \pi^{(D+1)} D \delta^{-2} e^{2C(1-\frac{D}{D_0}-\frac{1}{D})}}{\kappa^2 (\Gamma(\frac{D+1}{2}))^2} \\ &\times (D(D-1) \delta^{-2} e^{-2C/D} - 2\Lambda) \left(\frac{2D^2}{D_0} \right. \\ &\left. - \frac{D^2}{C} (\ln \pi - \psi(\frac{D+1}{2})) - \frac{2D}{C} - D - 1 \right). \end{aligned} \quad (112)$$

Let us now obtain the zero points of $U_I(D)$ and $U_{II}(D)$. Using the following asymptotic expression for Γ and ψ function as $D \rightarrow +\infty$

$$\Gamma(D) \rightarrow e^{-D} D^{D-\frac{1}{2}} \sqrt{2\pi} (1 + \frac{1}{12D} + \dots),$$

and

$$\psi(D) \rightarrow \ln D - \frac{1}{2D} - \sum_{k=0}^{\infty} \frac{B_{2k}}{2k D^{2k}},$$

where B_{2k} are the Bernoulli numbers, we obtain for $D \rightarrow +\infty$

$$U_I(D) \rightarrow \frac{2a_0^{2D_0}}{\kappa^2 \delta^4} (2\pi)^D D^{(4-D)} e^{D(1-\frac{2C}{D_0})} \rightarrow 0,$$

and

$$U_{II}(D) \rightarrow \frac{2a_0^{2D_0}}{\kappa^2 \delta^4 C} (2\pi)^D D^{(6-D)} e^{D(1-\frac{2C}{D_0})} \ln\left(\frac{D}{2}\right) \rightarrow 0.$$

Therefore, the potentials $U_I(D)$ and $U_{II}(D)$ both tend to zero as $D \rightarrow +\infty$. According to Eqs.(79-84) and using Eqs.(111,112) three other zero points of $U_I(D)$ and $U_{II}(D)$ are obtained. Those of $U_I(D)$ are

$$D = \begin{cases} D \rightarrow +\infty & \text{for } a \rightarrow \delta, \\ D_T & \text{for } a = a_T, \\ 1 & \text{for } a = a_{Im}, \\ 0 & \text{for } a \rightarrow +\infty. \end{cases} \quad (113)$$

For $U_{II}(D)$ we have the same zeros except the third one which is at $D = D_{IIIm}$ corresponding to $a = a_{IIIm}$, whose corresponding values are given in Table III.

We are now in a position to compare these potentials with the potential of the standard de Sitter minisuperspace depicted in Fig. 6. To make the comparison as simple as possible, we draw the dependence of our potential $U_I(a)$ against the scale factor a . As we see from Fig. 7, the behavior of the potential $U_I(a)$ for small scale factors of the order of the Planck length is similar to the potential $U(a)$. Instead, for large scale factors we see a completely different behavior. Here we have two

potential barriers with different heights depending on the value of C . For example, for $C = 1678$, corresponding to $D_{Pl} = 4$, the ratio of their heights is of the order of 10^{248} .

The height of the potential barrier in the region $\delta < a < a_T$ is also interesting to be compared to that of the potential $U(a)$. From Eq.(88), it is easy to see that the height of the potential barrier of $U(a)$, in the region $0 < a < l_\Lambda$, is of the order G^{-1} , or the square of the Planck energy, which is of the order of $10^{39} GeV^2$. In contrast, the height of the potential barrier of $U_I(a)$ in the region $\delta < a < a_T$ is about $10^{-158} GeV^2$ for $C = 1678$ and $10^{-2361} GeV^2$ for $C = 477$. Therefore as C increases, the height of the barrier increases so that for $C \simeq 10^6$ it is of the order G^{-1} comparable to the potential barrier $U(a)$. The behavior of $U_{II}(a)$ is similar to $U_I(a)$.

5.3 Wave function of the universe with dynamical space dimension

Here, we are interested in the solutions of the WDW equation in our model universe. Let us first use the semiclassical approximation to solve Eqs. (107) and then Eq.(108). The classically allowed region is $a_T < a < a_{Im}$, and the WKB solutions are

$$\Psi_{I\pm}(a) = [p_I(a)]^{-1/2} \exp[\pm i \int_{a_T}^a p_I(a') da' \mp i\pi/4], \quad (114)$$

where $p_I(a) = [-U_I(a)]^{1/2}$. Under the potential barrier in the region $\delta < a < a_T$ the WKB solutions are

$$\tilde{\Psi}_{I\pm}(a) = |p_I(a)|^{-1/2} \exp[\pm \int_a^{a_T} |p_I(a')| da']. \quad (115)$$

Finally, the solutions for the other under-barrier region $a > a_{Im}$ are

$$\hat{\Psi}_{I\pm}(a) = |p_I(a)|^{-1/2} \exp[\pm \int_{a_{Im}}^a |p_I(a')| da']. \quad (116)$$

From Eq.(102) we see that the classical momentum conjugate to a , p_{Ia} , is proportional to $-\dot{a}(D-1)$. For $a_T \ll a < a_{Im}$, Eq.(91) can be used for $\Psi_{I\pm}$:

$$\hat{p}_a \Psi_{I\pm}(a) \approx \pm p_I(a) \Psi_{I\pm}(a), \quad (117)$$

Thus $\Psi_{I-}(a)$ and $\Psi_{I+}(a)$ describe an expanding and a contracting universe, respectively. Now, we have to impose boundary condition in order to specify the wave function of our model uniquely. We choose, in accordance with the tunneling and no-boundary boundary condition, the following boundary condition as the scale factor tends to infinity:

$$\lim_{a \rightarrow +\infty} \Psi_I(a) \rightarrow 0. \quad (118)$$

Another boundary condition must be imposed at the turning point $a = \delta$ where the semiclassical approximation is bound to be wrong. Taking into account the normalization of the wave function [41], we can write it as

$$\Psi_I(\delta) = 1. \quad (119)$$

This satisfies the regularity of the wave function, i.e. $|\Psi| < \infty$, as in the case of Hartle–Hawking [28] and Vilenkin [31] boundary conditions. According to (118) the growing exponential $\hat{\Psi}_{I+}(a)$ should be absent under the barrier in the region $a > a_{Im}$:

$$\Psi_I(a > a_{Im}) = \hat{\Psi}_{I-}(a). \quad (120)$$

In the classically allowed region, the wave function is found by using the WKB connection formula at the turning point $a = a_{Im}$ [40]:

$$\begin{aligned} & \Psi_I(a_T < a < a_{Im}) \\ &= 2[p_I(a)]^{-1/2} \cos\left(\int_a^{a_{Im}} p_I(a') da' - \frac{\pi}{4}\right). \end{aligned} \quad (121)$$

This may also be written as

$$\begin{aligned}
\Psi_I(a_T < a < a_{Im}) &= 2[p_I(a)]^{-1/2} \\
&\times \sin\left(\int_{a_T}^{a_{Im}} p_I(a') da' - \int_{a_T}^a p_I(a') da' + \frac{\pi}{4}\right) \\
&= 2[p_I(a)]^{-1/2} \left\{ \sin\left(\int_{a_T}^{a_{Im}} p_I(a') da'\right) \right. \\
&\times \cos\left(\int_{a_T}^a p_I(a') da' - \frac{\pi}{4}\right) - \cos\left(\int_{a_T}^{a_{Im}} p_I(a') da'\right) \\
&\times \left. \sin\left(\int_{a_T}^a p_I(a') da' - \frac{\pi}{4}\right) \right\}. \tag{122}
\end{aligned}$$

Using (114), this gives

$$\begin{aligned}
\Psi_I(a_T < a < a_{Im}) &= \Psi_{I-}(a) e^{i\left(\int_{a_T}^{a_{Im}} p_I(a') da' - \frac{\pi}{2}\right)} \\
+ \Psi_{I+}(a) e^{-i\left(\int_{a_T}^{a_{Im}} p_I(a') da' - \frac{\pi}{2}\right)}. \tag{123}
\end{aligned}$$

Note that the dynamical character of the space dimension makes the wave function in the classical region a mixture of expanding and contracting part. It is in contrast to the standard case of constant space dimension where we can choose the wave function to be just expanding as assumed in the Vilenkin's tunneling boundary condition [30]. The presence of the expanding and contracting component in the classical region is due to the special form of our potential (see Fig. 7). The under-barrier wave function is found by using the WKB connection formula at the turning point $a = a_T$:

$$\begin{aligned}
\Psi_I(\delta < a < a_T) &= |p_I(a)|^{-1/2} \left\{ \sin\left(\int_{a_T}^{a_{Im}} p_I(a') da'\right) \right. \\
&\times \exp\left(-\int_a^{a_T} |p_I(a')| da'\right) + \cos\left(\int_{a_T}^{a_{Im}} p_I(a') da'\right) \\
&\times \left. \exp\left(\int_a^{a_T} |p_I(a')| da'\right) \right\}. \tag{124}
\end{aligned}$$

From (115), this may be written as

$$\begin{aligned}
\Psi_I(\delta < a < a_T) &= \sin\left(\int_{a_T}^{a_{Im}} p_I(a') da'\right) \tilde{\Psi}_{I-}(a) \\
&+ \cos\left(\int_{a_T}^{a_{Im}} p_I(a') da'\right) \tilde{\Psi}_{I+}(a). \tag{125}
\end{aligned}$$

The wave function $\Psi_I(a)$ is schematically represented in Fig. 7. Similarly, we obtain the following wave functions for the potential $U_{II}(a)$:

$$\begin{aligned}
\Psi_{II}(\delta < a < a_T) &= \sin\left(\int_{a_T}^{a_{IIIm}} p_{II}(a') da'\right) \tilde{\Psi}_{II-}(a) \\
+ \cos\left(\int_{a_T}^{a_{IIIm}} p_{II}(a') da'\right) \tilde{\Psi}_{II+}(a), \tag{126}
\end{aligned}$$

$$\begin{aligned}
\Psi_{II}(a_T < a < a_{IIIm}) &= \Psi_{II-}(a) e^{i\left(\int_{a_T}^{a_{IIIm}} p_{II}(a') da' - \frac{\pi}{2}\right)} \\
+ \Psi_{II+}(a) e^{-i\left(\int_{a_T}^{a_{IIIm}} p_{II}(a') da' - \frac{\pi}{2}\right)}, \tag{127}
\end{aligned}$$

and

$$\Psi_{II}(a > a_{IIIm}) = \hat{\Psi}_{II-}(a). \tag{128}$$

The above wave functions are different from those of Vilenkin, Hartle–Hawking, or Linde. But, as we will see, there are similarities between our general wave functions given by (120, 123, 125) for U_I or (126, 127, 128) for U_{II} and those of Vilenkin, Hartle–Hawking, and Linde. We notice first that our wave functions are real for every dimension, even for the limiting case of $C \rightarrow +\infty$ or $D \rightarrow D_0$. Therefore, we conclude that our general wave functions are not of Vilenkin’s type which is complex. In the classically allowed region, Ψ_{HH} and Ψ_L are both a superposition of the expanding and contracting terms with equal coefficients. In contrast, in our general wave functions the coefficients of expanding and contracting components are different. Looking at our general wave functions, we see that both decreasing and increasing exponential terms are present in the non-classical range of $\delta < a < a_T$. These means that in general, in this region, we expect our wave functions are a superposition of Ψ_{HH} and Ψ_L . Requiring the decreasing term, $\tilde{\Psi}_{I+}$, in (125) to vanish we are led to:

$$\int_{a_T}^{a_{Im}} p_I(a') da' = (2n + \frac{1}{2})\pi, \quad (129)$$

where n is an integer number. This requirement makes the coefficients of the expanding and contracting part of our wave functions in the classical region to be equal. Hence, assuming the relation (129) our general wave functions behave as Ψ_{HH} in the region $\delta < a < a_{Im}$. Now, if we require the increasing term $\tilde{\Psi}_{I-}$ to vanish under the potential barrier in the region of $\delta < a < a_T$, we obtain

$$\int_{a_T}^{a_{Im}} p_I(a') da' = 2m\pi, \quad (130)$$

where m is an integer number. This leads to the following wave functions:

$$\Psi_I^{(m)}(\delta < a < a_T) \equiv \tilde{\Psi}_{I+}(a), \quad (131)$$

$$\Psi_I^{(m)}(a_T < a < a_{Im}) \equiv i[\Psi_{I+}(a) - \Psi_{I-}(a)], \quad (132)$$

$$\Psi_I^{(m)}(a > a_{Im}) \equiv \hat{\Psi}_{I-}(a), \quad (133)$$

which are similar to the Linde’s wave function. $\Psi_I^{(m)}$ behaves as Ψ_L in the range of $\delta < a < a_T$. But in the classical region, i.e. $a_T < a < a_{Im}$, there are differences to Ψ_L . We call $\Psi_I^{(m)}$ the modified Linde wave function.

Now, we are interested in the form of our wave function in the limiting case of constant space dimension, i.e. $C \rightarrow +\infty$. In Appendix B, we show that as $C \rightarrow +\infty$ the relevant integral behaves in the following way:

$$\lim_{C \rightarrow +\infty} \int_{a_T}^{a_{Im}} p_I(a') da' = \lim_{C \rightarrow +\infty} \frac{\pi a_0^3 e^{2C}}{2G^{3/2}} \rightarrow +\infty. \quad (134)$$

Substituting this limiting behavior in (123) and (125), we see that the relevant terms are not well-defined. But we may assume cases in which C accepts those values corresponding to the relation (129) or (130). It can easily be seen that in this limit $\tilde{\Psi}_{I-}(a)$, $\Psi_{I+}(a)$ and $\Psi_{I-}(a)$ tend to $\tilde{\Psi}_{-}(a)$, $\Psi_{+}(a)$, $\Psi_{-}(a)$, respectively. Now, taking the limit of C to infinity corresponding to $n \rightarrow +\infty$, our wave functions approach to the Hartle–Hawking one:

$$\lim_{n \rightarrow +\infty} \Psi_I^{(n)}(\delta < a < a_T) \equiv \Psi_{HH}(0 < a < l_\Lambda), \quad (135)$$

$$\lim_{n \rightarrow +\infty} \Psi_I^{(n)}(a_T < a < a_{Im}) \equiv \Psi_{HH}(a > l_\Lambda). \quad (136)$$

If the limiting behavior of C in the relation (130) corresponds to $m \rightarrow +\infty$, then we obtain the modified Linde wave function:

$$\lim_{m \rightarrow +\infty} \Psi_I^{(m)}(\delta < a < a_T) \equiv \Psi_L(0 < a < l_\Lambda), \quad (137)$$

$$\lim_{m \rightarrow +\infty} \Psi_I^{(m)}(a_T < a < a_{Im}) \equiv i[\Psi_{+}(a) - \Psi_{-}(a)]. \quad (138)$$

It should be noted that in this limiting case, the second barrier in the large scale factors is removed and our potential $U_I(a)$ tends to the standard potential $U(a)$. Similarly, we can show that the same is true for the wave function $\Psi_{II}(a)$.

We conclude that assuming the boundary condition (118) and using WKB approach we always have contracting and expanding terms in the classical region, i.e. Vilenkin wave function can not be derived from our general wave function. This is due to the form of our potential which has a second barrier at large scale factors, independent of the C -value. Therefore, we may say that the Vilenkin wave function is structurally unstable with respect to the variation of dimension.

5.4 The probability density

It is interesting to find the probability density in our model universe and compare it to its value of the de Sitter minisuperspace in 3-space. From Eq.(94), we can calculate the probability density

$$\mathcal{P}_I \sim \exp(-2 \int_{\delta}^{a_T} |p_I(a')| da'), \quad (139)$$

for the potential barrier of $U_I(a)$ in the region $\delta < a < a_T$. As before, we take $\Lambda = 3G^{-1}$. The integral can be calculated numerically for different values of C :

$$\mathcal{P}_I \sim \begin{cases} \exp(-2.72) & \text{if } C = 1258800, \\ \exp(-2.28 \times 10^{-60}) & \text{if } C = 1678.8. \end{cases} \quad (140)$$

The probability density decreases as C increases, i.e. as the height of the potential barrier increases. Consequently, in the limit $C \rightarrow +\infty$, corresponding to the constant space dimension, the probability density has its minimum value given by Eq.(94):

$$\lim_{C \rightarrow +\infty} \mathcal{P}_I = \mathcal{P} \sim \exp(-\frac{3\pi}{G\Lambda}) = \exp(-\pi). \quad (141)$$

This is the same as the tunnelling probability proposed by Vilenkin and Linde. There is recently a controversy between Hartle, Hawking, and Turok on one side and Vilenkin and Linde on the other side on the sign of the action in the probability density ([34]–[39], [45]). Our calculation of the probability leads to the proposal of Vilenkin and Linde. There is another barrier in the region $a_{Im} < a < +\infty$ for the potential U_I , as seen by Fig. 7. The probability density for this potential barrier:

$$\hat{\mathcal{P}}_I \sim \exp(-2 \int_{a_{Im}}^{+\infty} |p_I(a')| da'), \quad (142)$$

is less than the probability density of the potential barrier in small scale factor. This is due to the height of the barrier for large scale factor is much more than the height of the barrier in small scale factor, see Fig. 7. The probability density

$$\mathcal{P}_{II} \sim \exp(-2 \int_{\delta}^{a_T} |p_{II}(a')| da'), \quad (143)$$

for the potential barrier of $U_{II}(a)$ in the region $\delta < a < a_T$ and

$$\hat{\mathcal{P}}_{II} \sim \exp(-2 \int_{a_{IIm}}^{+\infty} |p_{II}(a')| da'), \quad (144)$$

for the potential barrier in the region $a_{IIm} < a < +\infty$ can similarly be calculated and leads to similar results like $U_I(a)$. The existence of this barrier for large scale factors and the behavior of the potential in that region has cosmological consequences which will be discussed in a forthcoming paper [24].

Assuming $\Lambda = 3G^{-1} \sim 10^{38} GeV^2$, the Euclidean action $S_E = (-\frac{3\pi}{G\Lambda})$ takes the value $(-\pi)$. This large value for Λ -term is estimated in modern theories of elementary particle. Astronomical observations

indicates that the cosmological constant is much less than this value. Assuming $\Lambda \leq 10^{-80} \text{GeV}^2$, the Euclidean action is very large and negative ($S_E \sim -10^{120}$ typically). This gives the probability density proportional to $\mathcal{P} \propto \exp(-10^{120})$. Therefore, the probability density in the de Sitter minisuperspace depends on the assumed value for the cosmological constant. From the above discussion we conclude that for a given value of the cosmological constant, the probability density in our model is much more than the probability density of the de Sitter minisuperspace in 3-space. Therefore, the creation of a closed Friedman universe with variable spatial dimension is much more probable than the creation of a closed de Sitter universe in 3-space.

So in our model, the universe tunnels from the first turning point, $a = \delta$ to the second turning point, $a = a_T$, expands up to the third turning point $a = a_{Im}$ (or a_{IIIm} for U_{II}). It may then starts tunneling through the potential barrier in the region $a > a_{Im}$ (or a_{IIIm}). Seeing the process classically, the point a_{Im} (or a_{IIIm}) may act as a turning point at which the universe begins to contract [42].

6 CONCLUDING REMARKS

The idea of the space dimension to be other than three is relatively old, at least since the work of Kaluza and Klein. But it seems that the variability of space dimension is a relative new idea. We accept it as a viable alternative and formulate a cosmological toy model based on a Lagrangian to look for its consequences. In our formulation, there is a constraint and a scalar parameter, C , accepting every positive value. The limiting case of $C \rightarrow +\infty$ corresponds to the standard model of constant space dimension. Taking the present spatial dimension, D_0 , to be equal to 3, and assuming the space dimension at the Planck length, D_{Pl} , to be about 3, or one of the values of 4, 10, or 25 coming from the superstring theories, the corresponding values of C are of the order $10^6, 10^3, 10^2, 10^2$, respectively. For $D_0 = 2$ as a fractal dimension for matter distribution in the universe, coming from cosmological considerations, we obtain different values of C corresponding to $D_{Pl} = 3, 4, 10$ or 25.

After a critical review of the previous works on the decrumpling universe model, we generalized the Hawking–Ellis action for the perfect fluid universe in 3-space to the case of variable space dimension. Using this generalized action for a perfect fluid, we have then formulated the action for a FRW universe filled by a perfect fluid and allowing the space dimension to be a dynamical scalar variable. We have seen that this generalization is not unique. In contrast to the earlier works, we have taken into account the dependence of the measure of the integral on the spatial dimension. The Lagrangian and the equations of motion and also the time evolution equation of the spatial dimension are then obtained.

Using the time evolution equation of the spatial dimension, we have obtained the classical turning points of our model universe. It turns out that the space dimension could have been up to infinity at the beginning of the expansion phase of the universe, where the universe has a minimum size equal to δ . This is the only singularity we encounter in our model.

We then turn to quantum formulation of our toy model and write down the Wheeler–De Witt equation for our universe. Imposing the appropriate boundary condition at $a \rightarrow +\infty$ and using the semiclassical approximation we have obtained the wave function of our model, which is real but more general than its standard counterparts. As a result, the Hartle–Hawking wave function is obtained for special values of the constant C . Other values of C lead to a modified Linde wave function. In the standard limit of constant space dimension we may have a general, Hartle–Hawking, or a modified Linde wave function, but that of Vilenkin is not obtained. This is due to the special shape of the potential which has always a second barrier for large scale factors and only vanishes in the limiting case of $C \rightarrow +\infty$. We have therefore concluded that the Vilenkin wave function is, in respect of varying the space dimension, structurally unstable.

Finally we have calculated the probability density for the potential barrier in small scale factor. In the limit of constant spatial dimension, this probability density approaches to its corresponding value of the de Sitter minisuperspace as suggested by Vilenkin and Linde. In general, the probability density for our toy model is larger than that of the standard quantum cosmology.

Our toy model shows that it is possible to formulate a Lagrangian for a cosmological model with dynamical space dimension. It removes the usual singularities in physical quantities like scale factor of the universe because the minimum scale factor of the universe is δ . It gives us a model of how a probable space dimension of 10 or 25 at the Planck length can be incorporated in the usual picture of $(1+3)$ -dimensional space time and what its consequences are. We are currently studying other cosmological consequences of the model, specially those which can be compared to observational data.

Acknowledgments

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APPENDIX A: Volume of spacelike sections

The D -dimensional space metric is defined as (see Eq. 8)

$$\begin{aligned} d\Sigma_k^2 &= d\chi^2 + F^2(\chi)[d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \dots \\ &+ \sin^2 \theta_1 \dots \sin^2 \theta_{D-2} d\theta_{D-1}^2], \end{aligned} \quad (145)$$

where

$$0 \leq \theta_i \leq \pi \quad i = 1, \dots, D-2 \quad 0 \leq \theta_{D-1} \leq 2\pi.$$

The radius $F(\chi)$ is expressed as

$$F(\chi) = \begin{cases} \sin \chi & \text{if } k = +1, 0 \leq \chi \leq \pi, \\ \chi & \text{if } k = 0, 0 \leq \chi < \infty, \\ \sinh \chi & \text{if } k = -1, 0 \leq \chi < \infty. \end{cases} \quad (146)$$

We now consider the different cases $k = 0, \pm 1$, separately:

i) $k = +1$:

V_D is the volume of a unit D -sphere, S^D :

$$\begin{aligned} V_D &= \int_0^\pi \sin^{D-1} \chi d\chi \int_0^{2\pi} d\theta_{D-1} \\ &\times \prod_{m=2}^{D-1, D>2} \int_0^\pi d\theta_{D-m} \sin^{m-1} \theta_{D-m} \\ &= \frac{2\pi^{(\frac{D+1}{2})}}{\Gamma(\frac{D+1}{2})}. \end{aligned} \quad (147)$$

ii) $k = 0$:

In this case V_D is infinite. Introducing a cut-off, χ_c , as a very large number, V_D has the form

$$\begin{aligned} V_D &= \int_0^{\chi_c} \chi^{D-1} d\chi \int_0^{2\pi} d\theta_{D-1} \\ &\times \prod_{m=2}^{D-1, D>2} \int_0^\pi d\theta_{D-m} \sin^{m-1} \theta_{D-m} \\ &= \frac{\chi_c^D \pi^{D/2}}{\Gamma(\frac{D}{2} + 1)}. \end{aligned} \quad (148)$$

iii) $k = -1$:

Again, using the case of χ_C , we obtain

$$V_D = \int_0^{\chi_c} \sinh^{D-1} \chi d\chi \int_0^{2\pi} d\theta_{D-1}$$

$$\begin{aligned}
& \times \prod_{m=2}^{D-1, D>2} \int_0^\pi d\theta_{D-m} \sin^{m-1} \theta_{D-m} \\
& = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} f(\chi_c).
\end{aligned} \tag{149}$$

where

$$f(\chi_c) = \begin{cases} \frac{1}{2^{2m-1}} \sum_{k=0}^m (-1)^k \binom{2m}{k} \frac{\sinh(2m-2k)\chi_c}{2m-2k}, & \text{if } D = 2m+1, \\ \frac{1}{2^{2(m-1)}} \sum_{k=0}^{m-1} (-1)^k \binom{2m-1}{k} \frac{\cosh(2m-1-2k)\chi_c - 1}{2m-1-2k}, & \text{if } D = 2m. \end{cases} \tag{150}$$

where m is an integer number. For details of the about integration see [46]. In general χ_C tends to infinity so that for $k = 0, -1$, the volume of the spacelike sections are infinite. It should be emphasized that the above expression for V_D is valid for integer dimension. Volume of fractal structures having a non-integer dimension is beyond the scope of this paper.

APPENDIX B: Limiting behavior of our wave function for $C \rightarrow +\infty$

Our goal is to obtain the behavior of integrals in the wave function (123) and (125) as $C \rightarrow +\infty$:

$$\lim_{C \rightarrow +\infty} \int_{a_T}^{a_{Im}} p_I(a) da.$$

We calculate it by expanding $p_I(a)$ in inverse powers of C :

$$p_I(a) = \frac{3\pi a}{2G} \sqrt{\left(\frac{a^2}{G} - 1\right)} + \frac{f(a)}{C} + \mathcal{O}\left(\frac{1}{C^2}\right) + \dots, \tag{151}$$

where we take $D_0 = 3$ and

$$f(a) = \frac{27\pi a}{4G} \left(\ln \frac{a}{a_0}\right) \left\{ \frac{5}{6} \left(-1 + \frac{a^2}{G}\right)^{-1/2} - \left(-1 + \frac{a^2}{G}\right)^{1/2} \left[\frac{29}{6} + \ln \pi - \psi(2) \right] \right\}, \tag{152}$$

where $\psi(2) \simeq 0.42$. From Table III, we know that for $C \sim 10^6$ or so, the value of a_T is of the order of the Planck length. Therefore, we can also expand a_T in inverse powers of C :

$$a_T = \sqrt{G} + \frac{A}{C} + \mathcal{O}\left(\frac{1}{C^2}\right) + \dots, \tag{153}$$

where A is a constant. From Eqs.(151)–(153) we obtain

$$\int_{a_T}^{a_{Im}} p_I(a) da = \frac{\pi}{2} \left(-1 + \frac{a_{Im}^2}{G}\right)^{3/2} + \frac{A_1}{C} + \mathcal{O}\left(\frac{1}{C^2}\right) + \dots, \tag{154}$$

where A_1 is a constant. As previously mentioned, the relation between a_{Im} and C is

$$a_{Im} = a_0 e^{\frac{2C}{3}}. \tag{155}$$

Here a_0 is the scale factor corresponding to $D = D_0$. Inserting (155) into (154), we are led to

$$\lim_{C \rightarrow +\infty} \int_{a_T}^{a_{Im}} p_I(a) da = \lim_{C \rightarrow +\infty} \frac{\pi}{2} \left(-1 + \frac{a_0^2 e^{\frac{4C}{3}}}{G}\right)^{3/2} \sim \lim_{C \rightarrow +\infty} \frac{\pi a_0^3 e^{2C}}{2G^{3/2}}. \tag{156}$$

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FIGURE CAPTIONS

FIG. 1. The values of C as a function of D_{Pl} when $4 < D_{Pl} < 26$ and D_0 taking the values 2.9 (dashed line), 3.0 (solid line), 3.1 (dotted line). Using Eq.(6) it is easy to show that if we take $D_0 = 3$ for $D_{Pl} = 3$ and $D_{Pl} \rightarrow +\infty$, we have $C \rightarrow +\infty$ and $C = D_0 \ln \frac{H_0^{-1}}{l_{Pl}} \simeq 419.7$, respectively.

FIG. 2. The values of C as a function of $\log D_{Pl}$ when $3.6 < \log D_{Pl} < 6.2$ and D_0 taking the values 2.9999 (dashed line), 3.0000 (solid line), 3.0001 (dotted line).

FIG. 3. The values of $\log |\log \frac{\delta}{l_{Pl}}|$ as a function of D_{Pl} when $D_{Pl} \simeq 3$ and D_0 taking the values 2.990 (dashed line), 2.995 (solid line), 3.000 (dotted line).

FIG. 4. The values of $\log(\frac{\delta}{l_{Pl}})$ as a function of D_{Pl} when $4 < D_{Pl} < 26$ and D_0 taking the values 3.1 (dashed line), 3.0 (solid line), 2.9 (dotted line). Using Eq.(7), it is easy to show that if we take $D_0 = 3$, for $D_{Pl} = 3$ and $D_{Pl} \rightarrow +\infty$, we have $\log \frac{\delta}{l_{Pl}} \rightarrow -\infty$ and $\log \frac{\delta}{l_{Pl}} = 0$, respectively.

FIG. 5. The generic shape of the potential $V_{II}(D)$ for $k = +1$. The kinetic energy, \dot{D}^2 , is positive in the classical region, $D_{II m} < D < D_T$. The other values of D are corresponding to non-classical region in which the kinetic energy is negative.

FIG. 6. The potential $U(a)$ is shown by a solid line. The wave function of Hartle–Hawking (dashed curve) and that of Vilenkin (dotted curve) are shown. The real and imaginary parts of Vilenkin tunneling wave function are so indicated. The Hartle–Hawking wave function is real.

FIG. 7. The generic shape of the potential $U_I(a)$. The growing wave function $\tilde{\Psi}_{I-}$ (dashed line) and the decreasing wave function $\tilde{\Psi}_{I+}$ (dotted line) are shown. The under-barrier wave function, in the region $\delta < a < a_T$ is the sum of $\tilde{\Psi}_{I-}$ and $\tilde{\Psi}_{I+}$ (see Eq.125). In the region $a_T < a < a_{Im}$ there is an oscillating wave function. For $a > a_{Im}$ the decreasing wave function $\tilde{\Psi}_{I-}$ is also indicated. This shape is drawn approximately and it shows only the behavior of the potential $U_I(a)$. The similar shape can be drawn for the potential $U_{II}(a)$ and its corresponding wave function, except for a_{Im} instead of a_{Im} .

FIG. 1

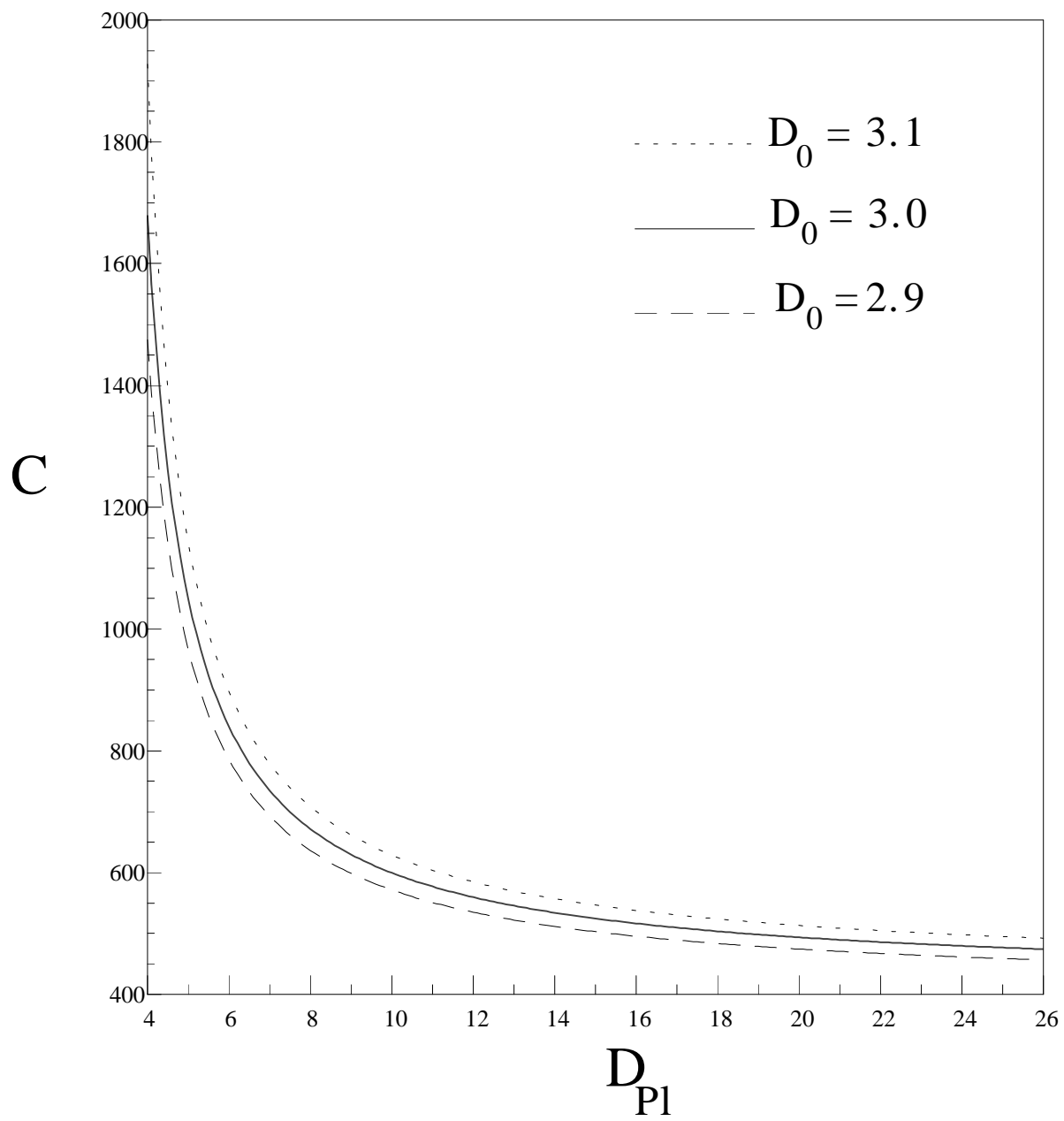


FIG. 2

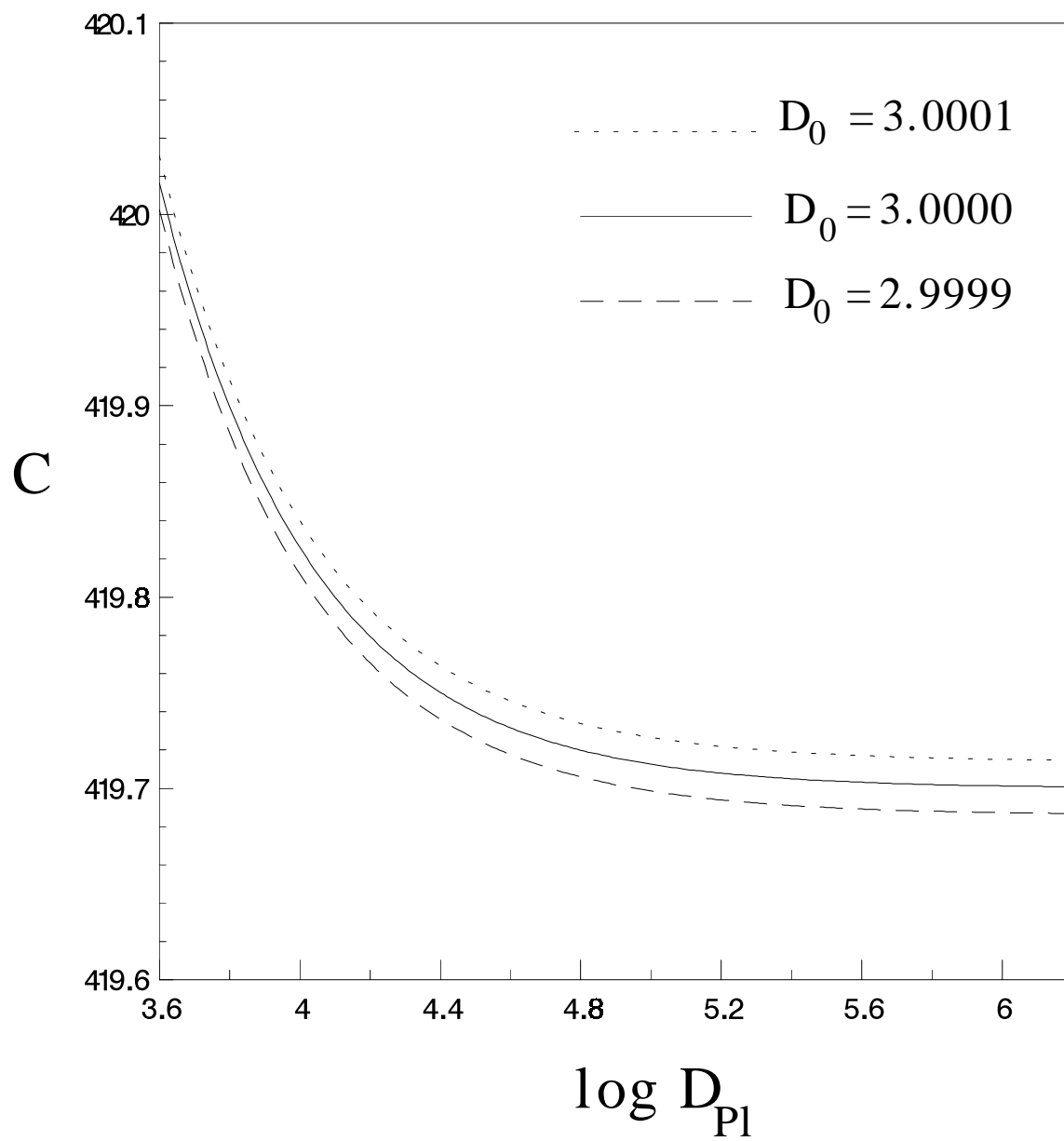


FIG.3

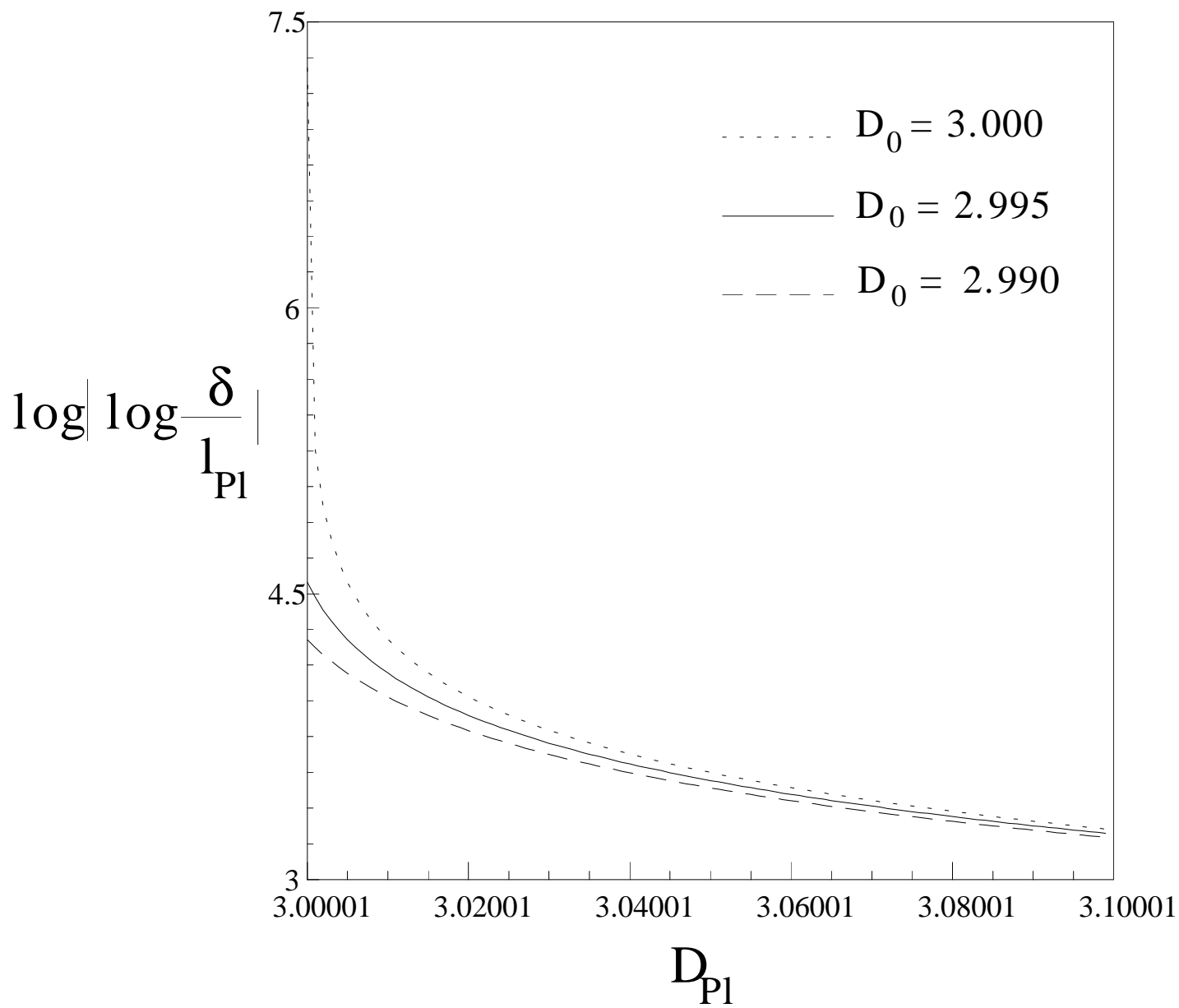


FIG.4

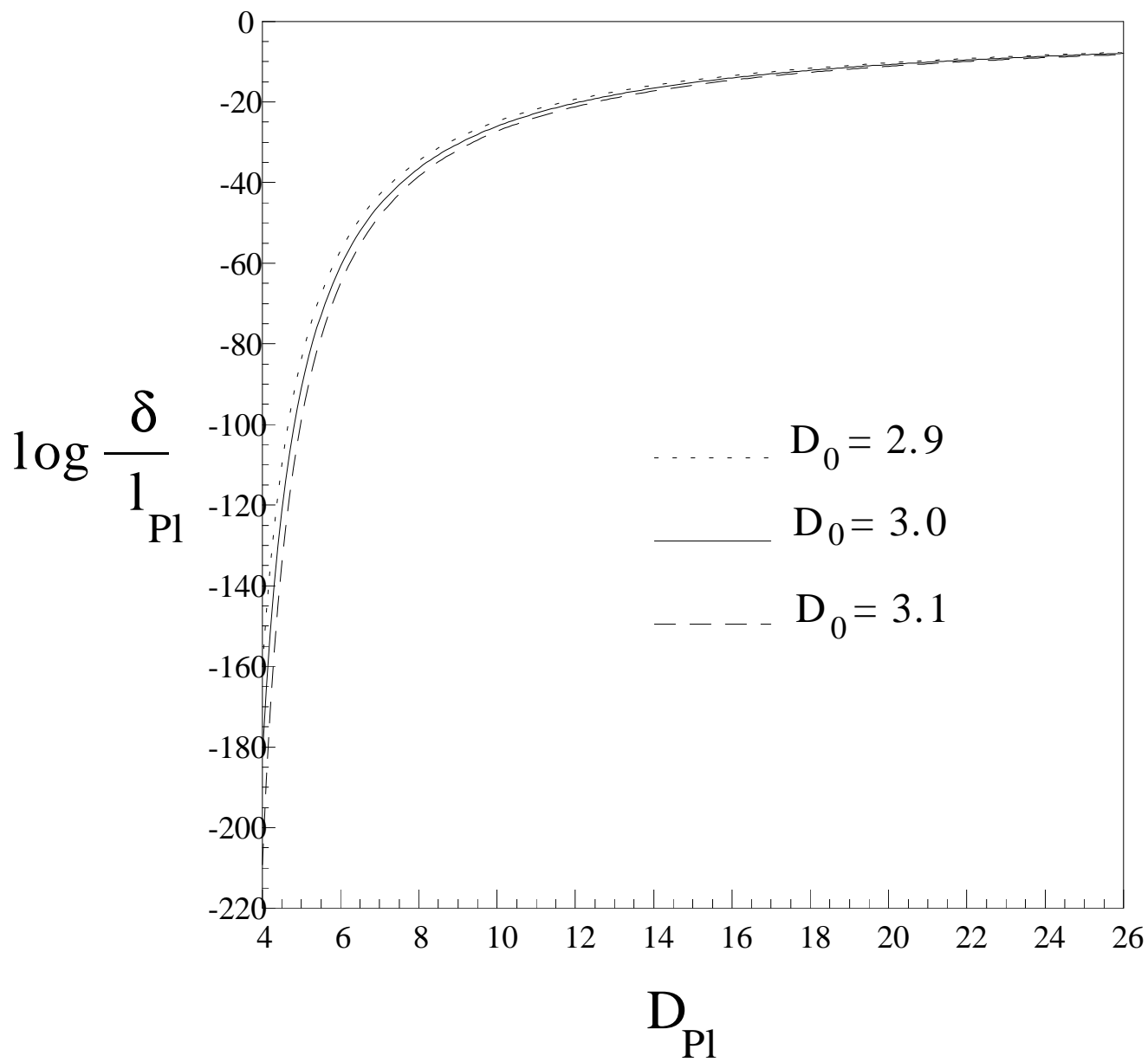


FIG. 5

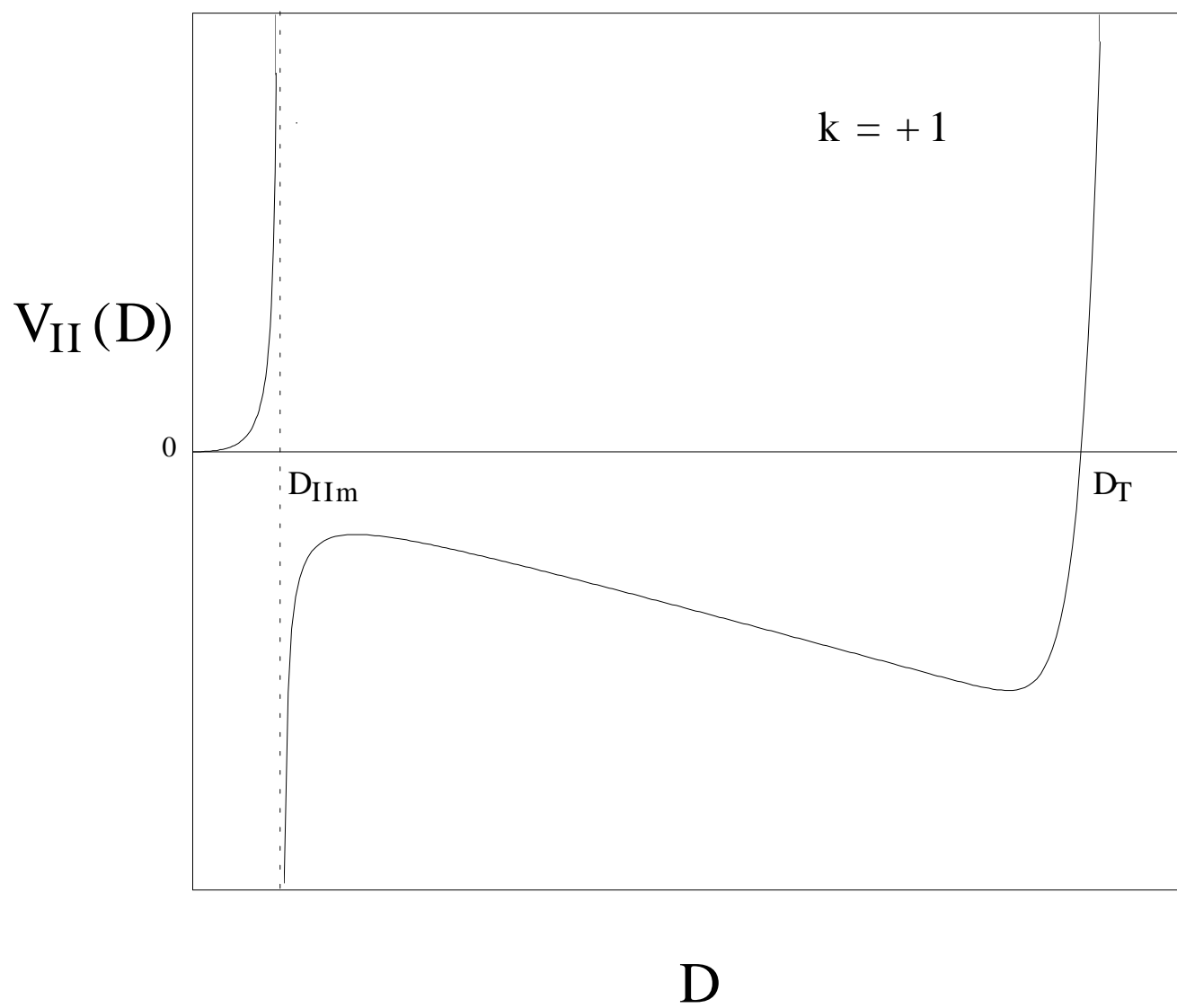


FIG. 6

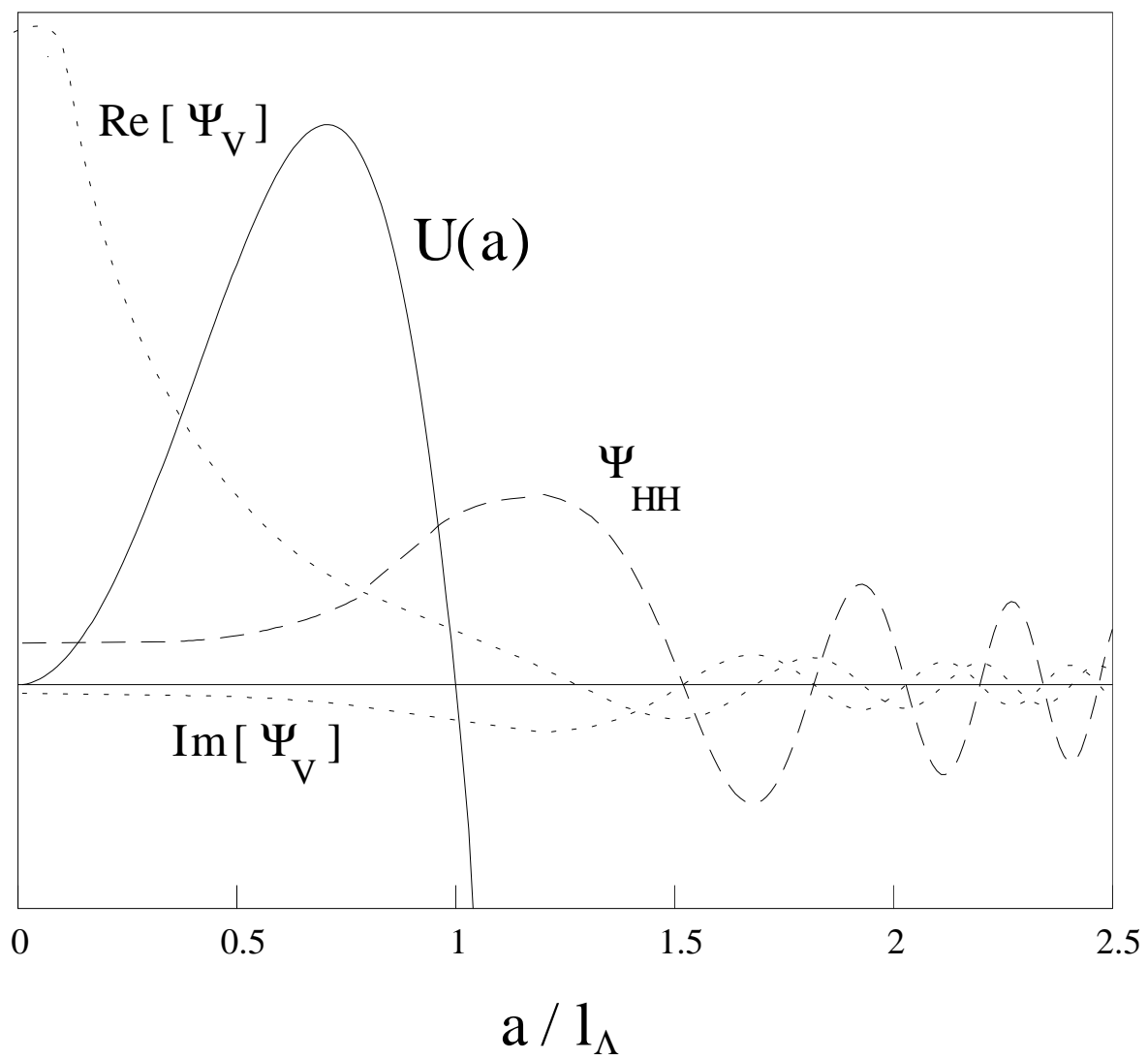


FIG. 7

